Lyapunov Stability

- Lyapunov stability presents a theory for stability of eq. pts.
  - other kinds of stability include, input-output stability, stability of periodic orbits.

- Stable eq. pt. if trajectories starting near eq. pt. remain near the eq. pt.
  eq. pt. asymptotically stable if trajectories converge to eq. pt.

- Lyapunov results provide sufficient conditions for stability / asymptotic-stability.
  - Also exist separate necessary conditions (called converse stability results).
  - Extension of Lyapunov results due to Lasalle is also to be discussed.

- Consider $\dot{x} = f(x)$ with eq. at 0, (and unique solution)
  (otherwise $y = x - x_{eq} \Rightarrow \dot{y} = \dot{x} = f(x) = f(y + x_{eq}) = g(y)$, with $y_{eq} = 0$)

  0 is stable if $\forall \varepsilon\ \exists \delta: \|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon,\ \forall t \geq 0$
  asymptotically stable if stable and for some $\delta: \|x(0)\| < \delta \Rightarrow x(t) \xrightarrow[t\to\infty]{} 0$.
  unstable if not stable.

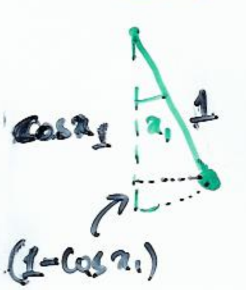
- Example: $\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a \sin x_1 - b x_2 \end{aligned}\right\}$ pendulum eq. $(l = 1, m = 1, g = a)$

  No friction $(b=0) \Rightarrow (0,0)$ is center $\Rightarrow$ (start close to eq. pt. $\Rightarrow$ stay close to eq. pt.)
  $\Rightarrow (0,0)$ stable
  But $(0,0)$ not asymptotically stable.

  Presence of friction $(b>0) \Rightarrow (0,0)$ stable focus $\Rightarrow$ asymptotically stable.
  $(\pi, 0)$ is saddle node $\Rightarrow$ unstable (trajectory starting near $(\pi,0)$ need not remain near $(\pi,0)$).

- Def. of stability implicity assumes existence of solution; this is guaranteed by Lyapunov's sufficient condition.

- Role of energy fn: $E(x) = \frac{1}{2} x_2^2 + a(1 - \cos x_1)$



$$\Rightarrow \dot{E} = x_2 \dot{x}_2 + a \sin x_1\, \dot{x}_1 = -x_2 (a \cancel{\sin x_1} + b x_2) + a \cancel{\sin x_1}\, x_2$$
$$= -b x_2^2$$

No friction $\Rightarrow \dot{E} = 0 \Rightarrow E = $ const. $\Rightarrow x$ does not grow (if $x(0) \approx 0$).
with friction $\Rightarrow \dot{E} < 0 \Rightarrow E \xrightarrow[t\to\infty]{} 0 \Rightarrow x \xrightarrow[t\to\infty]{} 0$ (if $x(0)$ near 0)

Lyapunov Stability (ctnd.)

- Lyapunov showed more general functions can be used for proving stability:
- Consider $V : D \to \mathbb{R}$ ($D \subseteq \mathbb{R}^n$ and $0 \in D$) such that (V called Lyapunov fn)
  - (i) V cont. diff
  - (ii) $V > 0$ ($\Leftrightarrow V(x) > 0$ for all $x \in D - \{0\}$ and $V(0) = 0$) +ve definite
  - (iii) $\dot{V} \leq 0$ ($\Leftrightarrow \dot{V}(x) \leq 0$ for all $x \in D$) −ve semi-definite

  Then 0 is stable.

  - (iii') $\dot{V} < 0$ ($\Leftrightarrow \dot{V}(x) < 0$ for all $x \in D - \{0\}$ and $\dot{V}(0) = 0$). −ve definite

  Then 0 is asymp. stable.

Sketch of proof: Given $\varepsilon > 0$, choose $r \in (0, \varepsilon]$ s.t. $B_r = \{x \mid \|x\| \leq r\} \subset D$. (possible since D open connected).

$\alpha := \min_{\|x\| = r} V(x) \Rightarrow \alpha > 0$ (from ii)

Pick $\beta \in (0, \alpha)$ and $\Omega_\beta := \{x \in B_r \mid V(x) \leq \beta\}$

$\Rightarrow \Omega_\beta$ in interior of B_r

(otherwise $\exists\, p \in \Omega_\beta \cap \partial B_r \Rightarrow V(p) \geq \alpha > \beta$. But $V(p) \leq \beta$ since $p \in \Omega_\beta$)

from (iii), Ω_β inv. since $x(0) \in \Omega_\beta \Rightarrow V(x(0)) \leq \beta$

Also $\dot{V} \leq 0 \Rightarrow V(x(t)) \leq \beta \Rightarrow x(t) \in \Omega_\beta$.

Also, Ω_β is closed (by def.) and bounded (contained in B_r which is bounded), i.e., Ω_β is compact, Follows from Thm 3.3 that $\dot{x} = f(x)$ has unique solution for all $t \geq 0$ whenever $x(0) \in \Omega_\beta$. (f is assumed to be locally Lipschitz over D)

Since V is cont. over D, and $V(0) = 0$, $\exists\, \delta : \|x\| \leq \delta \Rightarrow V(x) < \beta$.

Then $B_\delta \subset \Omega_\beta \subset B_r$, and $x(0) \in B_\delta \Rightarrow x(t) \in \Omega_\beta \subset B_r$, i.e.,

$\exists\, \delta : \|x(0)\| \leq \delta \Rightarrow \|x(t)\| \leq r \leq \varepsilon$. (proves stability of 0).

Asymptotic stability requires: $\forall a\ \exists T : \|x(t)\| < a$ for all $t \geq T$.

From previous argument, $\forall B_a \subset D\ \exists\, \Omega_b \subset B_a$. To show $\exists T$:

$\therefore$ $x(t) \in B_a$ for all $t \geq T$, suffices to show $x(t) \in \Omega_b$ for all $t \geq T$.

This in turn requires $V(x(t)) \xrightarrow[t \to \infty]{} 0$, which implies $\exists T : V(x(t)) \leq b$ for all $t \geq T$

$\Leftrightarrow x(t) \in \Omega_b$ for all $t \geq T$.

V bdd below by zero and $\dot{V} < 0 \Rightarrow V(x(t)) \xrightarrow[t \to \infty]{} 0$

Remark: $\dot{V}(x(t)) = \nabla V^T \dot{x} = \nabla V^T f$, i.e., (iii) implicitly requires cont. diff. of $V(x)$.

Lyapunov Stability (ctnd.)

Example: $\dot{x}_1 = x_2$

$\dot{x}_2 = -a\sin x_1 - b x_2$

Try $V(x) = E(x) = \frac{1}{2}x_2^2 + a(1-\cos x_1)$ as Lyapunov fn. candidate $\Rightarrow V(0)=0,\ V(x)>0$

$\Rightarrow \dot{V}(x) = \nabla V^T f = [a\sin x_1 \;\; x_2]\begin{bmatrix} x_2 \\ -a\sin x_1 - b x_2 \end{bmatrix} = \cancel{x_2 a\sin x_1} - \cancel{x_2 a \sin x_1} - b x_2^2$

$= -b x_2^2 \le 0$

so cannot prove asymptotic stability using above $V(x)$.

Try $V(x) = \frac{1}{2}x^T P x + a(1-\cos x_1) \Rightarrow \nabla V^T = (Px)^T + [a\sin x_1 \;\; 0]$

$\Rightarrow V(0)=0$ & $V>0 \iff P>0$

$\iff p_{11}>0,\ p_{11}p_{22} - p_{12}^2 > 0$

$= [p_{11}x_1 + p_{12}x_2 + a\sin x_1 \;\; \underbrace{p_{21}}_{=p_{12}\ (\text{since } P=P^T)}x_1 + p_{22}x_2]$

$\Rightarrow \dot{V}(x) = (p_{11}x_1 + p_{12}x_2 + a\sin x_1)x_2 - (p_{12}x_1 + p_{22}x_2)(a\sin x_1 + b x_2)$

To have $(a\sin x_1)x_2 - p_{22}x_2(a\sin x_1) = 0$, we choose $p_{22} = 1$

Also $p_{11}x_1x_2 - p_{12}b x_1 x_2 = 0$ when $p_{11} = p_{12}b$.

} make cross-terms zero which are sign indefinite

$\Rightarrow \dot{V}(x) = p_{12}x_2^2 - b x_2^2 - p_{12}x_1 a \sin x_1$.

Also, $p_{11}p_{22} - p_{12}^2 = p_{12}b - p_{12}^2 = p_{12}(b-p_{12}) > 0 \iff 0 < p_{12} < b$

If we choose $p_{12} = \frac{b}{2}$, $\dot{V}(x) = -\frac{b}{2}x_2^2 - \frac{ab}{2}x_1\sin x_1$

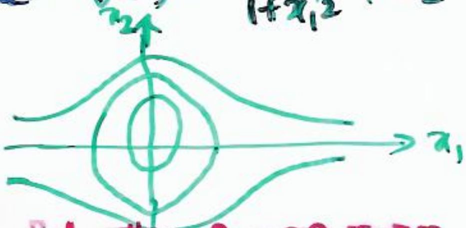
Since $x_1\sin x_1 > 0$ for $0<|x_1|<\pi$, we can choose $D = \{x \mid |x_1| < \pi\}$.

Then the origin is asymptotically stable.

Region of Stability

- Suppose $x=0$ is asymptotically stable and $\Phi(t;x)$ is solution of $\dot{x} = f(x)$ starting at x.

  ROS $= \{x \mid \Phi(t;x) \to 0 \text{ as } t\to\infty\}$; $x=0$ globally asymp. stable if ROS $= \mathbb{R}^n$
- Lyapunov fn. may be used to estimate ROS. From Thm 4.1, if $\Omega_c = \{x \mid V(x) \le c\}$ is bounded and contained in D (domain over $V(x)>0$ & $\dot{V}(x)<0$), then $\Omega_c \subseteq$ ROS.
- For any c, Ω_c need not be bounded. Example: $V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$.



For $\Omega_c \subseteq B_r$ need: $c < \underbrace{\inf_{\|x\|=r} V(x)}$ inf value $V(x)$ can take on boundary of B_r

$c < l = \underbrace{\lim_{r\to\infty}\ \inf_{\|x\|=r} V(x)} < \infty \Rightarrow \Omega_c$ bounded.

(work out) inf value $V(x)$ can take on boundary of B_r as $r\to\infty$

$\min_{\|x\|=r}\left(\frac{x_1^2}{1+x_1^2} + x_2^2\right) = \frac{r}{1+r} \xrightarrow[r\to\infty]{} 1 \Rightarrow \Omega_{<1}$ is bounded.

Lyapunov Stability (ctnd.)

Thm 4.2 Consider system with $f(0)=0$, and $V:\mathbb{R}^n \to \mathbb{R}$ cont. diff. s.t. $V>0$, $\dot V<0$, and $\underbrace{\|x\| \to \infty \Rightarrow V(x)\to\infty}$. Then 0 globally asymptotically stable. (Barbashin – Krasovskii thm).

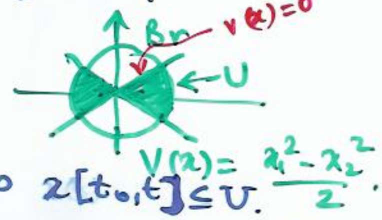
→ Radial unboundedness condition

Proof: $\forall c\ \exists r : \|x\| > r \Rightarrow V(x) > c$.

$\Leftrightarrow \forall c\ \exists r : V(x) \le c \Rightarrow \|x\| \le r \Leftrightarrow \forall c \exists r : \Omega_c \subseteq B_r$.

Note: For 0 to be globally asymp. stable, it must be unique pt.
So global asymp. stability studied for only single eq. pt. systems.
(pendulum system does not qualify)

Thm 4.3 (Unstability) System with $f(0)=0$, and cont. diff. $V: D \to \mathbb{R}$ s.t. $V(0)=0$ and $\exists r: B_r \subseteq D$, and $\forall \rho < r\ \exists x \in B_\rho : V(x) > 0$. Let $U = \{x \in B_r \mid V(x) > 0\}$. If $\dot V(x) > 0$ for all $x \in U$, then "0" is unstable.



Proof: Pick $x_0 \in U \Rightarrow V(x_0) = a > 0$

Since $\dot V(x) > 0$ for all $x \in U \Rightarrow V(x(t)) \ge a$ as long as $x[t_0,t] \subseteq U$.

Let $\gamma = \min\{\dot V(x) \mid x \in U \text{ and } V(x) \ge a\} = \min\{\dot V(x) \mid \underbrace{x \in B_r \text{ and } V(x) \ge a}_{\text{compact set}}\}$

$\dot V = V_x f$ is cont. since V_x cont. and f locally Lipschitz $\Rightarrow f$ cont.

So γ exists (min of a cont. fn. over a compact set exists). Then $\gamma > 0$.

Also, $V(x(t)) = V(x_0) + \int_{t_0}^{t} \dot V(x(s))\,ds \ge a + \int_{t_0}^{t} \gamma\, ds = a + \gamma(t - t_0)$, grows arbitrarily

Since $V(x(t))$ bounded on $\bar U$ ($V(x)$ is cont. on $\bar U$, and $\bar U$ is compact),
$x(t)$ must eventually leave U. Since $V(x(t)) \ge a$, $x(t)$ cannot leave U through the surface $V(x)=0$, i.e., $x(t)$ leaves U through $\|x\|=r \Rightarrow$ unstable eq. pt.

Example:
$$\left.\begin{aligned} \dot x_1 &= x_1 + g_1(x) \\ \dot x_2 &= -x_2 + g_2(x) \end{aligned}\right\}$$
g_1, g_2 locally Lipschitz and $|g_1(x)|, |g_2(x)| \le k\|x\|_2^2$.

Let $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$. Then $V(x) > 0$ for $x_1 \ne 0$, $x_2 = 0$.

$$\dot V(x) = [x_1 \quad -x_2]\begin{bmatrix} x_1 + g_1(x) \\ -x_2 + g_2(x)\end{bmatrix} = x_1^2 + x_1 g_1(x) + x_2^2 - x_2 g_2(x) = \|x\|_2^2 + [x_1 g_1(x) - x_2 g_2(x)]$$

$|x_1 g_1(x) - x_2 g_2(x)| \le \sum |x_i||g_i(x)| \le 2k\|x\|_2^3 \Rightarrow \dot V(x) \ge \|x\|_2^2 - 2k\|x\|_2^3$

Choose $r : r < \frac{1}{2k} \Rightarrow \dot V(x) > 0$ for all $x \in B_r$. Thus "0" is unstable.