Fundamental Properties

- Fundamental properties of solution of $\dot{x} = f(t, x)$, x_0 given.
  - existence & uniqueness, continuous dependence on initial condition / parameter

- Model is used for prediction of behavior $\Rightarrow$ existence & uniqueness is important
- Initial condition/model may be imprecise $\Rightarrow$ cont. dependence on their value is important

Background: $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}^+$ s.t. (i) $\|x\| = 0 \Leftrightarrow x = 0$

(ii) $\|x + y\| \leq \|x\| + \|y\|$

(iii) $\|\alpha x\| = |\alpha| \, \|x\|$

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \quad 1 \leq p < \infty \; ; \quad \|x\|_\infty = \max_{i=1}^{n} |x_i|$$

All p-norms are equivalent: $c_1 \|x\|_\beta \leq \|x\|_\alpha \leq c_2 \|x\|_\beta$

$\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\, \|x\|_2$; $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\, \|x\|_\infty$; $\|x\|_\infty \leq \|x\|_1 \leq n \|x\|_\infty$

Hölder's inequality: $|x^T y| \leq \|x\|_p \|y\|_q$, $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \|x\|_2 \leq \sqrt{\|x\|_p \|x\|_q}$

Induced norm of matrix: $\|A\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \max_{\|x\|_p = 1} \|Ax\|_p$

$$\|A\|_1 = \max_j \underbrace{\left(\sum_{i=1}^{m} |a_{ij}| \right)}_{\text{row-sum}} \; ; \quad \|A\|_2 = \sqrt{\lambda_{max}(A^T A)} \; ; \quad \|A\|_\infty = \max_i \underbrace{\left(\sum_{j=1}^{n} |a_{ij}| \right)}_{\text{column-sum}}$$

$\frac{1}{\sqrt{n}} \|A\|_\infty \leq \|A\|_2 \leq \sqrt{m}\, \|A\|_\infty$; $\frac{1}{\sqrt{m}} \|A\|_1 \leq \|A\|_2 \leq \sqrt{n}\, \|A\|_1$; $\|AB\|_p \leq \|A\|_p \|B\|_p$

Continuity: $f : \mathbb{R}^n \to \mathbb{R}^m$ continuous at $x \in \mathbb{R}^n$ if
$\forall \varepsilon > 0 \; \exists \delta > 0 : \|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < \varepsilon$.

f cont. in $D \subseteq \mathbb{R}^n$ if f cont. at each $x \in D$.

f uniformly cont. in $D \subset \mathbb{R}^n$ if the same δ can be chosen for all $x \in D$.

uniform cont. $\Rightarrow$ cont., converse may not hold. It holds when D compact.

f_1 cont., f_2 cont. $\Rightarrow$ $f_1 \circ f_2$ cont.

f cont., D compact $\Rightarrow$ $f(D)$ compact. $\Rightarrow$ cont. fn. on compact set are bdd.

$\Rightarrow \exists\, p, q \in D$: $f(q) \leq f(x) \leq f(p)$.

f cont., D connected $\Rightarrow$ $f(D)$ connected.

f cont. and 1-to-1, D compact $\Rightarrow$ f^{-1} exist, is cont. over $f(D)$.

Background

Convergence: $\{x_k\}$ converges to x if $\forall \varepsilon\ \exists N: \|x_k - x\| < \varepsilon\ \forall k \geq N$, in which case x is called limit point.

- x is an accumulation point if a subseq. of $\{x_k\}$ converges to x.
- x is sup.(inf.) limit pt. if it is sup.(inf.) accumulation pt.
- A monotonically increasing (decreasing), bdd from above (below) seq. of reals converges.
- $\{x_k\}$ Cauchy if $\forall \varepsilon\ \exists N: \|x_k - x_m\| < \varepsilon\ \forall k, m \geq N$. Convergent $\Rightarrow$ Cauchy (converse may not hold)

Sets: $D \subset \mathbb{R}^n$ open if $\forall x \in D\ \exists \varepsilon > 0: \|y - x\| < \varepsilon \Rightarrow y \in D$.

- $D \subset \mathbb{R}^n$ closed if $(\mathbb{R}^n - D)$ open $\Leftrightarrow$ every convergent seq. has limit pt in D.
- $D \subset \mathbb{R}^n$ bounded if $\exists M > 0,\ \forall x \in D: \|x\| \leq M$.
- $D \subset \mathbb{R}^n$ compact if D closed and bounded.
- $x \in D$ is boundary pt. if $\forall \varepsilon > 0\ \exists y \in D, z \in D^c: \|x - y\|, \|x - z\| < \varepsilon$
- ∂D set of boundary pts. of D. D closed iff $\partial D \subseteq D$.
- Interior of $D = D - \partial D$; Closure of D, $\bar{D} = D \cup \partial D$.
- D connected if $x, y \in D \Rightarrow \exists$ arc connecting x & y in D.
- D convex if D connected and arc can be chosen to be line.
- D domain if open & connected

Differentiability: $f: \mathbb{R} \to \mathbb{R}$ differentiable at x if $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} =: f'(x)$ exists

f continuously diff at x if $f'(x)$ exists and $f'(x)$ continuous at x.

$f: \mathbb{R}^n \to \mathbb{R}^m$ diff. (cont. diff.) at x if $\dfrac{\partial f_i}{\partial x_j}$ exists (and is cont.) at x.

Mean value thm: $f: \mathbb{R}^n \to \mathbb{R}$ cont. diff. at all points in open set D containing a line $L(x, y) \subseteq D$. Then $\exists z \in L(x, y)$: $\dfrac{f(y) - f(x)}{y - x} = \dfrac{\partial f}{\partial x}\Big|_{x = z}$

Implicit fn. thm: $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ cont. diff. in open set D, with $f(x_0, y_0) = 0$, $\dfrac{\partial f}{\partial x}(x_0, y_0) \neq 0 \Rightarrow \exists$ open $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ with $x_0 \in U$ and $y_0 \in V$,

$g: V \to U$ s.t. $g(y) = x \Leftrightarrow f(x, y) = 0$, and g cont. diff. at y_0.

Banach Space & Contraction Mapping Thm: Banach space $\equiv$ Complete Normed Vector Space $\to$ $\boxed{\text{Cauchy} \Rightarrow \text{Convergent}}$

S closed subset of Banach space, and $f: S \to S$ s.t. $\|f(x) - f(y)\| \leq \rho \|x - y\|$, $0 < \rho < 1$

$\exists\ x^* \in S$ s.t. $f(x^*) = x^*$

$\forall x \in S$, $\{f^k(x)\}$ converges to x^*.

Consider $\{x_k = (\frac{1}{2})^k\}$ over $\mathbb{R} - \{0\}$. Then $\{x_k\}$ Cauchy but not convergent.

$$\begin{array}{ccccc} \text{cont. diff.} & \Rightarrow & \text{diff.} & \Rightarrow & \text{cont.} \\ \Downarrow & & & & \Uparrow \\ \text{Lipschitz} & \Rightarrow & \text{locally Lipschitz} & & \end{array}$$

$x^{1/3}$: diff, not cont diff., not locally Lipschitz at 0

Existence and uniqueness

- Consider $\dot{x} = x^{1/3}$ and $x(0) = 0$

  Then two possible solutions are: $x(t) = 0$ and $x(t) = \left(\frac{2t}{3}\right)^{3/2}$ $\Rightarrow$ nonunique

  Such a model is not very useful since it does not predict future uniquely.

- Existence of unique solution requires notion of Lipschitz condition.

(i) $f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ locally Lipschitz at (t_0, x_0) if

$\exists\, \varepsilon, \delta,\ \forall x, y \in B_\varepsilon(x_0),\ t_0 - \delta \leq t \leq t_0 + \delta$: $\quad \|f(t,x) - f(t,y)\| \leq L\|x - y\|$

(ii) $f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ locally Lipschitz over $[a,b] \times D$ (D open & connected)

if (i) holds for every $(t, x) \in [a,b] \times D$

(iii) Further, f is Lipschitz over $[a,b] \times D$ if "L" is chosen to be same uniformly.

(iii) $f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ globally Lipschitz if Lipschitz over $[0,\infty) \times \mathbb{R}^n$.

Note: Locally Lipschitz over $[a,b] \times D$ $\Rightarrow$ Lipschitz over compact subset of $[a,b] \times D$.

Thm 1: $\exists\, \delta > 0$: $\dot{x} = f(t,x)$ has unique solution over $[t_0, t_0 + \delta]$ if

f locally Lipschitz at $(t_0, x(t_0))$ and piece-wise cont. in t over $[t_0, t_0+\delta]$

(proof based on contraction mapping theorem)

Sufficient cond. 1: $f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ cont. over $D \subset \mathbb{R}^n$, diff. over $[a,b] \times D$

For $[a,b]$, convex $W \subset D$, $\exists\, L$: $\left\|\frac{\partial f}{\partial x}\right\| \leq L$ $\Rightarrow$ f locally Liptshitz over $[a,b] \times W$.

$$f(x) = x^{1/3} \Rightarrow \frac{\partial f}{\partial x} = \frac{1}{3} x^{-2/3} \xrightarrow[x \to 0]{} \infty \Rightarrow \text{infinite slope at } 0 \Rightarrow \text{not Lipschitz.}$$

Necessary cond. 1: f locally Lipschitz over $[a,b] \times D \Rightarrow f$ cont. over $[a,b] \times D$.

Suff. cond. 2: f cont. & cont. diff. over $[a,b] \times D \Rightarrow f$ locally Lipschitz over $[a,b] \times D$

Suff 3. f cont. & cont. diff over $[a,b] \times D \Rightarrow$ [f globally Lipschitz over $[a,b] \times D$ $\Leftrightarrow$ $\frac{\partial f}{\partial x}$ uniformly bounded on $[a,b] \times D$].

Example

$$f(x) = \begin{bmatrix} -x_1 + x_1 x_2 \\ x_2 - x_1 x_2 \end{bmatrix}$$ cont. & cont. diff. on $\mathbb{R}^2$ $\Rightarrow$ locally Lipschitz on $\mathbb{R}^2$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 + x_2 & x_1 \\ x_2 & 1 - x_1 \end{bmatrix}$$ not uniformly bdd on $\mathbb{R}^2$ $\Rightarrow$ not (globally) Lipschitz on $\mathbb{R}^2$

But Liptschitz on a compact set $D \subseteq \mathbb{R}^2$, say $\{|x_1| < a_1, |x_2| < a_2\}$.

$$\Rightarrow \|\partial f / \partial x\| = \max\{|-1 + x_2| + |x_1|,\ |x_2| + |1 - x_1|\} \leq \underbrace{1 + a_1 + a_2}_{= L}$$

Proving Existence of Unique Solution (Thm 3.1 & 3.2)

- $\dot{x} = f(t, x) \Rightarrow x(t) = \underbrace{x_0 + \int_{t_0}^{t} f(s, x(s))\, ds}_{(Px)(t)}$

  $\Rightarrow x$ a cont. fn. of t, i.e. $x \in \underbrace{C[t_0, t_0+\delta]}_{\text{set of cont. fns over } [t_0, t_0+\delta]}$ for all $t_0 + \delta \geq t_0$ for which $x(t)$ defined

- We can view $(Px)(t) : C[t_0, t_0+\delta] \to C[t_0, t_0+\delta]$ (maps $x(\cdot)$ to $x(t)$)

  Then $x(t) = (Px)(t)$, and $x(t)$ is fixed point of this map.

- To prove uniqueness contraction mapping thm can be applied.

  First note that $C[t_0, t_0+\delta]$ is a Banach space under norm, $\|x[t_0, t_0+\delta]\| = \max_{t \in [t_0, t_0+\delta]} \|x(t)\|$

  Need to choose closed set $S \subset C[t_0, t_0+\delta]$ and δ so that P is contraction over S.

- Let $S := \{x(\cdot) \in C[t_0, t_0+\delta] \mid \|x(\cdot) - x_0\| \leq r\} \Rightarrow S$ closed.

- Need to show P maps S to S, i.e., $x(\cdot) \in S \Rightarrow (Px)(\cdot) \in S$, i.e, need to show $\|(Px)(\cdot) - x_0\| \leq r$

$$(Px) - x_0 = \int_{t_0}^{t_0+\delta} f(s, x(s))\, ds = \int_{t_0}^{t_0+\delta} [f(s, x(s)) - f(s, x_0) + f(s, x_0)]\, ds$$

f piece-wise cont. in $t \Rightarrow f(t, x_0)$ bounded over $[t_0, t_0+\delta]$; $h := \max_{t \in [t_0, t_0+\delta]} \|f(t, x_0)\|$

$$\Rightarrow \|P(x) - x_0\| \leq \int_{t_0}^{t_0+\delta} (\|f(s, x(s)) - f(s, x_0)\| + h)\, ds$$

$$\leq \int_{t_0}^{t_0+\delta} (L\|x(s) - x_0\| + h)\, ds \leq \int_{t_0}^{t_0+\delta} (Lr + h)\, ds = \delta(Lr + h)$$

Thus we need $\delta(Lr+h) \leq r \Leftrightarrow \boxed{\delta \leq \frac{r}{Lr+h}}$

- Next need to show P is a contraction over S.

$$\|(Px) - (Py)\| = \left\| \int_{t_0}^{t_0+\delta} [f(s, x(s)) - f(s, y(s))]\, ds \right\| \leq \int_{t_0}^{t_0+\delta} L\|x(s) - y(s)\|\, ds$$

$$\leq \int_{t_0}^{t_0+\delta} L\|x(\cdot) - y(\cdot)\|\, ds = \delta L \|x(\cdot) - y(\cdot)\|$$

For contraction, need $\delta L \leq \rho < 1 \Leftrightarrow \boxed{\delta \leq \rho/L}$

So choose $\delta \leq \min\left\{\frac{\rho}{L}, \frac{r}{Lr+h}\right\} \Rightarrow$ From contraction mapping, unique solution exists over $[t_0, t_0+\delta]$ in S.

- Since the solution cannot leave S within $[t_0, t_0+\delta]$, the solution is also unique over $C[t_0, t_0+\delta]$.

Thm 3.2 : L is global $\Rightarrow$ choose r, ρ s.t. $\frac{\rho}{L} < \frac{r}{Lr+h} \Leftrightarrow r(1-\rho) > \frac{h}{L}$ $\Bigg\{$ So $\delta \leq \frac{\rho}{L}$ works above; Divide $[t_0, t_1]$ into finite # of δ intervals and apply Thm 3.1