Invariance Principle

- In pendulum example, $\dot{x}_1 = x_2$, $\dot{x}_2 = -a\sin x_1 - bx_2$, $V(x) = \frac{1}{2}x_2^2 + a(1-\cos x_1)$

is not adequate to show asymptotic convergence since $\dot{V}(x) = -\frac{1}{2}bx_2^2 \leq 0$, which implies $\dot{V}(x) = 0$ whenever $x_2 = 0$ ($\dot{V}$ is not -ve definite).

- However, $\dot{V}(x) = 0 \Rightarrow x_2 = 0 \Rightarrow \dot{x}_2 = 0 \Rightarrow \sin x_1 = 0 \Rightarrow x_1 = 0$ (assuming $|x_1| < \pi$).
So when $x_1 \neq 0$, $V(x)$ must decrease. (This is expected in presence of friction.)
- If exists $V(x)$ with $\dot{V}(x) \leq 0$ around origin, and $\dot{V}(x) = 0 \Rightarrow x = 0$, then origin must be asymp. stable. (known as Lasalle's Invariance Principle)

Definitions: p the limit pt. of $x(t)$ if $\exists \{t_n\}$ s.t. $\{x(t_n)\} \xrightarrow[n\to\infty]{} p$

Set of all +ve limit pts called +ve limit set.

M +vely inv. if $x(0) \in M \Rightarrow x(t) \in M \;\; \forall t \geq 0$

$x(t)$ approaches M ($x(t) \to M$) if $\forall \varepsilon > 0 \; \exists T :$ $\underbrace{\|x(t), M\|}_{\inf_{y \in M}\|x(t) - y\|} < \varepsilon \;\; \forall t \geq T$.

Example: Stable eq. pt. +ve limit pt. of points near the eq. pt.
Stable limit cycle +ve limit set of points near the limit cycle
Also $x(t)$ approaches stable eq. pt. or stable limit cycle.
eq. pt. & limit cycle are invariant sets
$\Omega_c = \{x \in \mathbb{R}^n \mid V(x) \leq c \;\&\; \dot{V}(x) \leq 0\}$ is +vely inv. set.

Lemma: $x(t)$ bounded and contained in D for $t \geq 0$, then $x(t)$ has a +ve-limit set L^+ that is nonempty, compact and invariant. Also $x(t) \xrightarrow[t\to\infty]{} L^+$.

Inv. Principle: $V: D \to \mathbb{R}$ cont. diff, $\Omega \subset D$ compact, +vely-inv. s.t. $\dot{V}(x) \leq 0$ on Ω.
$E = \{x \in \Omega \mid \dot{V}(x) = 0\}$ and $M \subset E$ largest inv. set. Then $x(0) \in \Omega \Rightarrow x \xrightarrow[t\to\infty]{} M$.

Pick $x(0) \in \Omega \Rightarrow x(t) \in \Omega$ and $V(x(t))$ decreases monotonically
$V(x)$ cont. on Ω, Ω compact $\Rightarrow V(x)$ is lower bounded $\Big\} \Rightarrow V(x(t)) \xrightarrow[t\to\infty]{} a$

Also from Lemma, $\exists L^+$ s.t. L^+ nonempt, compact, inv. Further $L^+ \subset \Omega$ since Ω is closed (and so contains all limit points),
$\forall p \in L^+ \; \exists \{t_n\}$ s.t. $\{x(t_n)\} \xrightarrow[n\to\infty]{} p \;\underset{V \text{ cont.}}{\Rightarrow}\; \{V(x(t_n))\} \xrightarrow[n\to\infty]{} V(p) = a$
$\Rightarrow V(p) = a$ for all $p \in L^+$. L^+ inv. $\Rightarrow x(0) \in L^+ \Rightarrow x(t) \in L^+ \Rightarrow \dot{V}(x) = 0$ for $x \in L^+$
$\Rightarrow L^+ \subset E \subset \Omega$. M largest inv. subset of $E \Rightarrow L^+ \subset M \subset E \subset \Omega$. So, $x(t) \to L^+$ $\Rightarrow x(t) \to M$.

Invariance Principle (ctnd.)

- In the Inv. theorem, $V(x) > 0$ not required.
- Also Ω is not necessarily based on V. In many applications, V itself provides Ω, e.g., $\Omega_c = \{x \mid V(x) \le c\}$ may be bounded and $\dot{V}(x) \le 0$ over Ω_c. Then choose $\Omega = \Omega_c$.
- Also $V > 0 \Rightarrow \exists\, c > 0$ s.t. Ω_c is bounded (not necessarily true always). For radially unbounded V, Ω_c is bounded for all c.

Corollary: $V: D \to \mathbb{R}$ cont. diff., +ve-definite over $D \ni 0$ s.t. $\dot{V}(x) \le 0$ in D. $E = \{x \in D \mid \dot{V}(x) = 0\}$ is such that no solution can stay in E except $x(t) \equiv 0$. Then origin is asymp. stable. (M (largest inv. subset of E) $= \{0\}$)

Corollary: $V: \mathbb{R}^n \to \mathbb{R}$ cont. diff., radially unbounded, +ve-definite s.t. $\dot{V}(x) \le 0$. $E = \{x \in \mathbb{R}^n \mid \dot{V}(x) = 0\}$ is such that no solution can stay in E except $x(t) \equiv 0$. Then origin globally asymp. stable. (M (largest inv. subset of E) $= \{0\}$)

Example (generalized pendulum)

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -h_1(x_1) - h_2(x_2) \quad \Rightarrow \quad \boxed{V(x) = \int_0^{x_1} h_1(y)\,dy + \frac{1}{2} x_2^2}$$
$$h_i(0) = 0, \; y\, h_i(y) > 0 \;\; \forall y \ne 0, \; y \in (-a, a)$$

$$\dot{V}(x) = h_1(x_1)\, x_2 + x_2\left[-h_1(x_1) - h_2(x_2)\right] = -x_2 h_2(x_2) \le 0$$

$\dot{V}(x) = 0 \Rightarrow x_2 h_2(x_2) = 0 \Rightarrow x_2 = 0$ (since $x_2 h_2(x_2) > 0 \;\; \forall\, x_2 \ne 0,\; x_2 \in (-a, a)$)

Thus $E = \{x \in D \mid \dot{V}(x) = 0\} = \{x_2 = 0\}$. Also $x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow h(x_1(t)) \equiv 0 \Rightarrow x_1(t) \equiv 0$

The only solution that can stay in E is 0. From Lasalle's thm, 0 asymp. stable.

Suppose $a = \infty$ and additionally, $\int_0^{x_1} h_1(y)\,dy \xrightarrow[|x_1| \to \infty]{} \infty \Rightarrow V(x)$ radially unbounded

Also it can be shown that $\dot{V}(x) \le 0$ for $x \in \mathbb{R}^2$, and $E = \{x \in \mathbb{R}^2 \mid \dot{V}(x) = 0\} = \{x_2 = 0\}$ contains only the trivial solution $x(t) \equiv 0$. Follows that origin is globally asymp. stable.

Lasalle's Thm:
1) Relaxes the requirement that $\dot{V}(x) < 0$
2) Region of attraction can be approximated as Ω, a set with form different from $\Omega_c = \{x \mid V(x) \le c\}$.
3) Does not require existence of isolated eq. pt. (can be eq. set)
4) $V(x)$ need not be > 0.

Invariance Principle Example

<u>Example:</u> 1^{st} order system $\begin{cases} \dot{y} = ay + u \\ u = -ky \,, \quad \dot{k} = \gamma y^2 \,, \quad \gamma > 0 \end{cases}$

$x_1 = y \,,\ x_2 = k \Rightarrow \begin{cases} \dot{x}_1 = a x_1 - x_2 x_1 \\ \dot{x}_2 = \gamma x_1^2 \end{cases}$

At equilibrium, $(a - x_2) x_1 = \gamma x_1^2 = 0 \Rightarrow \{x_1 = 0\}$ is equilibrium set

To show that trajectory approaches the set $x_1 = 0$ (adaptive controller regulates output to zero), let $V(x) = \frac{1}{2} x_1^2 + \frac{1}{2\gamma} (x_2 - b)^2$, $b > a$.

$\dot{V}(x) = x_1 \dot{x}_1 + \frac{1}{\gamma} (x_2 - b) \dot{x}_2 = (a - x_2) x_1^2 + (x_2 - b) x_1^2 = -x_1^2 (b - a) \le 0$

Also $V(x)$ is radially unbounded $\Rightarrow \forall c \, \exists r : \Omega_c \subset B_r$ and Ω_c compact & +vely inv.

$E = \{x \in \Omega_c \mid \underbrace{x_1 = 0}_{\Rightarrow \dot{V}(x) \le 0}\}$. This set is inv. since it is an eq. set. So $M = E$.

From Lasalle's inv. thm, trajectories starting in Ω_c approach E for any c.

<u>Note:</u> $V(x)$ has parameter b which need not be explicitly known, i.e., it may be possible to have existence of a desired $V(x)$ without explicitly knowing it.

Linear systems & Linearizations

$\dot{x} = Ax$ has isolated eq. at 0 if $\det A \ne 0$ (in general, eq. set = null space of A).

Stability property of a linear system can be characterized using locations of eigen values

$\dot{x} = Ax \Rightarrow x(t) = e^{At} x(0)$. Let $P^{-1} A P = J = \text{block diag}[J_1 \ J_2 \dots J_r]$, where

J_i is Jordan block associated with eigen-value λ_i of A, $J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}_{m_i \times m_i}$, of order m_i.

$\Rightarrow e^{At} = P e^{Jt} P^{-1} = \sum_{i=1}^{r} \sum_{k=1}^{m_i} t^{k-1} e^{\lambda_i t} R_{ik}$

If λ_i has multiplicity q_i, then Jordan blocks associated with λ_i are all of order 1 iff $\text{rank}(A - \lambda_i I) = n - q_i$ (n is dimension of x).

<u>Thm:</u> $x = 0$ is stable eq. of $\dot{x} = Ax$ iff $\text{Re}(\lambda_i) \le 0$ and $\text{Re}(\lambda_i) = 0 \Rightarrow \text{rank}(A - \lambda_i I) = n - q_i$

$x = 0$ is globally asymp. stable iff $\text{Re}(\lambda_i) < 0$.

<u>Proof:</u> 0 is stable iff e^{At} bounded for $t \ge 0$. If $\text{Re}(\lambda_i) > 0 \Rightarrow e^{At}$ cannot be bounded and so we must have $\text{Re}(\lambda_i) \le 0$. If $\text{Re}(\lambda_i) = 0 \Rightarrow e^{At}$ cannot be bounded if $m_i \ge 2$.

So we must have $\text{Re}(\lambda_i) = 0 \Rightarrow \text{rank}(A - \lambda_i I) = n - q_i$. This establishes necessity.

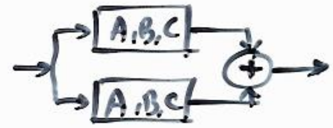
Sufficiency follows from $x(t) = P^{-1} e^{Jt} P \, x(0)$ and $e^{Jt} = \sum_{i=1}^{r} \sum_{k=1}^{m_i} t^{k-1} e^{\lambda_i t} R_{ik}$.

For asymp. stability, $e^{At} \xrightarrow[t \to \infty]{} 0$ is N&S. This holds iff $\text{Re}(\lambda_i) < 0$.

Linear system & Linearization (ctnd.)

Example.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$



$$y = [1 \quad 0]\, x$$

For the given system, $\lambda = \pm j \Rightarrow$ stable

For the series system, $A_s = \begin{bmatrix} A & 0 \\ \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} & A \end{bmatrix}$ and for parallel system, $A_p = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$

Then for A_s and A_p, $\lambda = \pm j$ with $q_i = 2$ $\quad (\lambda_1 = j, \lambda_2 = -j)$

Also, $\text{rank}[A_p - \lambda_i I] = n - q_i = 4 - 2 = 2$, $\text{rank}[A_s - \lambda_i I] = 3 \neq n - q_i$

Thus parallel connection stable, while series connection unstable.

In parallel connection, non-zero initial condition $\Rightarrow$ const. amplitude osc. in both copies of the system. Sum of constant amp. osc. of same freq. $\Rightarrow$ constant amp. osc.

In series connection, the const. amp. osc. of 1st copy, excites the 2nd copy. Since the 2nd copy has natural freq. of 1 rad/sec which is the freq. of driving input, "resonance" occurs and response grows unbounded.

A called Hurwitz if $\text{Re}(\lambda_i) < 0$ for all i.

Lyapunov method can be used to investigate asymp. stability of $\dot{x} = Ax$.

Consider $V(x) = x^T P x$ as a choice of Lyapunov fn. $V > 0 \Leftrightarrow P > 0$

Also $\dot{V}(x) = 2x^T P \dot{x} = \dot{x}^T P x + x^T P \dot{x} = x^T A^T P x + x^T P A x = x^T (A^T P + PA) x$

We want P s.t., $A^T P + PA = -Q$ for some $Q > 0$ to ensure asyp stability.

Thm: A is Hurwitz iff $\forall Q > 0 \;\exists P > 0 : \; A^T P + PA = -Q$.

($\Leftarrow$) Choose $V(x) = x^T P x$.

($\Rightarrow$) Set $P = \int_0^\infty e^{A^T t} Q e^{At}\, dt$. Since $e^{At} = \sum_{i=1}^{r} \sum_{k=1}^{m_i} t^{k-1} e^{\lambda_i t} R_{ik}$

and $\text{Re}(\lambda_i) < 0$, the integral exists and is finite.

To see $P > 0$, consider $x^T P x = \int_0^\infty (x^T e^{A^T t} Q e^{At} x)\, dt$ for $x \neq 0$.

Since $Q > 0$, $x^T e^{A^T t} Q e^{At} x > 0$ for all t $\Rightarrow \int_0^\infty (x^T e^{A^T t} Q e^{At} x)\, dt > 0$.

Finally, $A^T P + PA = \int_0^\infty [A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A]\, dt = \int_0^\infty \frac{d}{dt} [e^{A^T t} Q e^{At}]\, dt$

$$= e^{A^T t} Q e^{At} \Big|_{t=0}^{t=\infty} = 0 - Q = -Q.$$

Now if $\tilde{P}$ is another solution, i.e. $-Q = A^T \tilde{P} + \tilde{P} A$. Then,

$P = \int_\infty^0 (e^{A^T t} [A^T \tilde{P} + \tilde{P} A] e^{At})\, dt = \int_\infty^0 \frac{d}{dt} [e^{A^T t} \tilde{P} e^{At}]\, dt = e^{A^T t} \tilde{P} e^{At} \Big|_\infty^0 = \tilde{P} - 0 = \tilde{P}$.

Remark: Q can be chosen to be $C^T C$ ($\Rightarrow Q \geq 0$) such that (A, C) observable.

Linear system & Linearization (ctnd.)

Since Q can be chosen to be any +ve definite matrix, one choice is $Q = I$.

Example: $A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$.

$$\left.\begin{aligned} A^T P &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} p_{12} & p_{22} \\ -p_{12}-p_{11} & -p_{22}-p_{12} \end{bmatrix} \\ PA &= (A^T P)^T = \begin{bmatrix} p_{12} & -p_{12}-p_{11} \\ p_{22} & -p_{22}-p_{12} \end{bmatrix} \end{aligned}\right\} \Rightarrow A^T P + PA = \begin{bmatrix} 2p_{12} & -p_{12}+p_{22}-p_{11} \\ -p_{12}+p_{22}-p_{11} & 2(-p_{22}-p_{12}) \end{bmatrix} = -I$$

$$\Rightarrow \begin{cases} 2p_{12} = -1 \Rightarrow p_{12} = -\frac{1}{2} \\ 2(-p_{22}-p_{12}) = -1 \Rightarrow p_{22} = 1 \\ -p_{12}+p_{22}-p_{11} = 0 \Rightarrow 3/2 \end{cases} \Rightarrow P = \begin{bmatrix} 1.5 & -.5 \\ -.5 & 1 \end{bmatrix}. \quad \underbrace{\det(P) = 1.25 > 0 \ \& \ \det[1.5] > 0}_{\Rightarrow P > 0.}$$

Checking whether $Re(\lambda_i) < 0$ is easier than determining $P > 0$ such as above. The real advantage of finding P is in proving stability properties of linearization.

Given $\dot{x} = f(x)$ with $f(0) = 0$, $f(x) = Ax + g(x)$, where

$A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$ and $g_i(x) = \left[\frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)\right] x$. Note from mean-value theorem,

$$f_i(x) = f_i(0) + \frac{\partial f_i}{\partial x}(z_i)\, x = \frac{\partial f_i}{\partial x}(z_i) x, \text{ where } z_i \text{ lies on line from } 0 \text{ to } x.$$

Thm: Origin asymp. stable if $Re(\lambda_i) < 0 \ \forall \lambda_i(A)$ and unstable if $Re(\lambda_i) > 0$ for some $\lambda_i(A)$, where $A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$

For first part, use $V(x) = x^T P x$ as candidate. $V > 0 \Leftrightarrow P > 0$. Also,

$$\dot{V} = 2x^T P \dot{x} = x^T P \dot{x} + \dot{x}^T P x = x^T P f + f^T P x = x^T P[Ax + g] + [Ax+g]^T P x.$$
$$= x^T [PA + A^T P] x + 2x^T P g = -x^T Q x + 2x^T P g$$

$$\frac{\|g\|}{\|x\|} \xrightarrow[\|x\| \to 0]{} 0 \Rightarrow \forall \gamma > 0, \exists r > 0 : \ \|g\| \leq \gamma \|x\| \ \forall \|x\| < r.$$

So, $\dot{V} < -x^T Q x + 2\gamma \|P\| \|x\|^2$, $\forall \|x\| < r$. Also $x^T Q x \geq \lambda_{min}(Q) \|x\|^2$

$\Rightarrow \dot{V} < -[\lambda_{min}(Q) - 2\gamma \|P\|] \|x\|^2$. Thus choosing $\gamma < \frac{1}{2} \lambda_{min}(Q) / \|P\|$ ensures $\dot{V} < 0$.

For 2nd part, first suppose $Re(\lambda_i) \neq 0$ forall λ_i. By defining $z = Tx$, we can have $TAT^{-1} = \begin{bmatrix} -A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ with $A_1, A_2 > 0$ and $\begin{cases} \dot{z}_1 = -A_1 z_1 + g_1(z) \\ \dot{z}_2 = A_2 z_2 + g_2(z) \end{cases}$, $\forall \gamma > 0\ \exists r > 0 : \|g_i(z)\| \leq \gamma \|z\| \ \forall \|z\| \leq r$.

$A_i > 0 \Leftrightarrow Re(\lambda_i) > 0, \forall \lambda_i \Leftrightarrow \forall Q_i > 0 \ \exists P_i > 0 : P_i A_i + A_i^T P_i = -Q_i$.

Define $V(z) = z_1^T P_1 z_1 - z_2^T P_2 z_2 = z^T \begin{bmatrix} P_1 & 0 \\ 0 & -P_2 \end{bmatrix} z \Rightarrow V(z) > 0$ for $z_2 = 0$.

Define $U = \{ z \mid \|z\| \leq r \text{ and } V(z) > 0 \}$.

Linearization (ctnd.)

$$\dot{V}(z) = 2z_1^T P_1 \dot{z}_1 - 2z_2^T P_2 \dot{z}_2 = \dot{z}_1^T P_1 z_1 + z_1^T P_1 \dot{z}_1 + \dot{z}_2^T P_2 z_2 - z_2^T P_2 \dot{z}_2$$

$$= (-A_1 z_1 + g_1)^T P_1 z_1 + z_1^T P_1(-A_1 z_1 + g_1) - (A_2 z_2 + g_2)^T P_2 z_2 - z_2^T P_2 (A_2 z_2 + g_2)$$

$$= -z_1^T (A_1^T P_1 + P_1 A_1) z_1 + 2 z_1^T P_1 g_1 - z_2^T (A_2^T P_2 + A_2 P_2) z_2 - 2 z_2^T P_2 g_2$$

$$= z_1^T Q_1 z_1 + z_2^T Q_2 z_2 + 2 z^T \begin{bmatrix} P_1 g_1 \\ -P_2 g_2 \end{bmatrix}$$

$$\geq \lambda_{min}(Q_1) \|z_1\|^2 + \lambda_{min}(Q_2) \|z_2\|^2 - 2\|z\| \sqrt{\|P_1\|^2 \|g_1\|^2 + \|P_2\|^2 \|g_2\|^2}$$

$$> \alpha \underbrace{(\|z_1\|^2 + \|z_2\|^2)}_{\|z\|^2} - 2\|z\| \sqrt{2\beta^2 \gamma^2 \|z\|^2} \quad \left(\alpha = \min\{\lambda_{min}(Q_1), \lambda_{min}(Q_2)\}, \ \beta = \max\{\|P_1\|, \|P_2\|\}\right)$$

$$= (\alpha - 2\sqrt{2}\beta\gamma) \|z\|^2.$$

Thus choosing $\gamma < \frac{\alpha}{2\sqrt{2}\beta}$ ensures $\dot{V} > 0$. Unstability follows from Thm 4.3.

When $Re(\lambda_i) = 0$ for some λ_i, then let $\delta = \min_{\lambda_i : Re(\lambda_i) > 0} Re(\lambda_i) > 0$.

Then $(A - \frac{\delta}{2} I)$ is such that $Re(\lambda_i (A - \frac{\delta}{2} I)) \neq 0$. From previous analysis, for $Q > 0$ exists P s.t. $P[A - \frac{\delta}{2} I] + [A - \frac{\delta}{2} I]^T P = Q > 0$ and $V(x) = x^T P x$ is +ve for points arbitrarily close to 0.

Also, $\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = x^T [A^T P + PA] x + 2 x^T P g$

$$= x^T [(A - \tfrac{\delta}{2} I)^T P + P (A - \tfrac{\delta}{2} I)] x + \delta x^T P x + 2 x^T P g$$

$$= x^T Q x + \delta V(x) + 2 x^T P g(x)$$

In the set, $\{x \in \mathbb{R}^n \mid \|x\| \leq r \text{ and } V > 0\}$, where r is chosen so that $\|g\| < \gamma \|x\|$ for $\|x\| < r$, we have

$\dot{V} \geq \lambda_{min}(Q) \|x\|^2 - 2\gamma \|P\| \|x\|^2 = (\lambda_{min}(Q) - 2\gamma \|P\|) \|x\|^2$, which is +ve if $\gamma < \frac{\lambda_{min}(Q)}{2\|P\|}$. Unstability follows from Thm 4.3.

Remark: Stability property of nonlinear system can be deduced from its linearization provided $Re(\lambda_i) \neq 0$. The test is based on computation of $Re(\lambda_i)$ and checking its location, whereas proof is based on Lyapunov thms.