Centralized Control

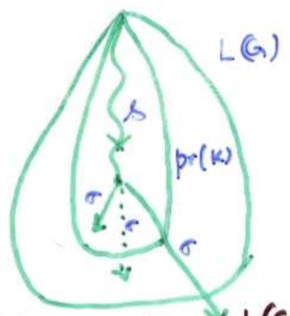
- Given G does there exist Σ_u-enabling and non-marking S s.t.

  either (i) $L(G\|S) = K$ (K is desired generated behavior)

  or (ii) $L_m(G\|S) = K$ and S is non-blocking (K is desired marked lang)

- Need notions of controllability and relative-closure:

- Controllability: K controllable if $pr(K)\Sigma_u \cap L(G) \subseteq pr(K)$



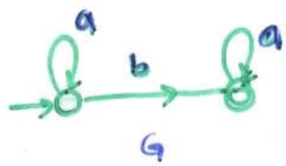
$s \in pr(K)$, $\sigma \in \Sigma_u$, $s\sigma \in L(G) \Rightarrow s\sigma \in pr(K)$

Note: S Σ_u-enabling iff $L(G\|S)$ is controllable.
K controllable iff $pr(K)$ controllable.

- Relative-closure: $pr(K) \cap L_m(G) = K \cap L_m(G)$ (K prefix-closed relative to $L_m(G)$)

$K \subseteq L_m(G) \Rightarrow [pr(K) \cap L_m(G) = K] \Leftrightarrow [pr(K) \cap L_m(G) \subseteq K]$.

- Example:



$L_m(G) = a^* b a^*$; $L(G) = pr(L_m(G))$
$\Sigma_u = \{b\}$.

$K = \{a^k b a^k \mid k \geq 0\}$ (equal a's preceding and following single b)

$s \in pr(K)$; $\left.\begin{array}{l}\text{if } s \in a^*, \text{ then } sb \in pr(K) \\ \text{if } s \notin a^*, \text{ then } sb \notin L(G)\end{array}\right\} \Rightarrow K$ controllable

$s = ab \in pr(K) \cap L_m(G) - K \Rightarrow K$ not relative-closed.

$K = \{a^k \mid k > n\} \cup pr[\{a^k b a^* \mid k \leq n\}]$ (block b after at most n initial a's)

$a^{n+1} \in pr(K)$, $a^{n+1} b \in L(G) - pr(K) \Rightarrow K$ not controllable.

Existence of Supervisor

Thm: Given G and $K \subseteq \Sigma^*$, exists Σ_u-enabling, non-marking S s.t.

$$L(G \| S) = K \quad \text{iff} \quad \phi \neq K = pr(K) \subseteq L(G) \text{ and } K \text{ controllable.}$$

Pf: ($\Rightarrow$) since $L(G\|S) = K$, we have $\phi \neq K = pr(K) \subseteq L(G)$

S Σ_u-enabling $\Rightarrow$ $L(G\|S) = K$ controllable

($\Leftarrow$) Choose S s.t. $L_m(S) = L(S) = K$
(can be done since $\phi \neq K = pr(K)$).

So $L(G\|S) = L(G) \cap L(S) = L(G) \cap K = K$ (since $K \subseteq L(G)$).

Also, K controllable $\Rightarrow$ $L(G\|S)$ controllable $\Rightarrow$ S Σ_u-enabling

Finally, $L_m(G\|S) = L_m(G) \cap L_m(S) = L_m(G) \cap L(S)$ $\Rightarrow$ S non-marking.

Example: 

G

$\Sigma_u = \{b\}$; $L_m(G) = a^* b a^*$; $L(G) = pr(a^* b a^*)$.

$K = pr\left[\{a^k b a^k \mid k \geq 0\}\right]$ is controllable, prefix closed, nonempty sublang of $L(G)$.

So, S with $L_m(S) = L(S) = K$ achieves the desired behavior K.

$K' = \{a^k \mid k > n\} \cup pr\left[\{a^k b a^* \mid k \leq n\}\right]$ not controllable

So desired S does not exist.

Existence of Supervisor

Thm: Given G and $K \subseteq \Sigma^*$, exists Σ_u-enabling, non-marking, non-blocking supervisor S s.t.
$L_m(G||S) = K$ iff $\emptyset \neq K = pr(K) \cap L_m(G)$, and K controllable.

Pf: ($\Rightarrow$) $L_m(G||S) = K$, S non-blocking $\Rightarrow$ $L(G||S) = pr\big(L_m(G||S)\big) = pr(K)$.

So $pr(K) \neq \emptyset \Rightarrow K \neq \emptyset$.

S Σ_u-enabling $\Rightarrow$ $L(G||S)$ controllable $\Rightarrow$ $pr(K)$ controllable $\Rightarrow$ K controllable.

Also, $pr(K) \cap L_m(G) = L(G||S) \cap L_m(G) = L(G) \cap L(S) \cap L_m(G) = L_m(G) \cap L(S)$

$$= L_m(G||S) = K$$

(since S non-marking)

($\Leftarrow$) Choose S s.t. $L_m(S) = L(S) = pr(K)$ $\Rightarrow$ S non-marking

(can be done since $K \neq \emptyset \Rightarrow pr(K) \neq \emptyset$).

$L_m(G||S) = L_m(G) \cap L_m(S) = L_m(G) \cap pr(K) = K$

$L(G||S) = L(G) \cap L(S) = L(G) \cap pr(K) = pr(K)$ $\quad$ ($K \subseteq L_m(G) \subseteq L(G)$ $\Rightarrow pr(K) \subseteq L(G)$).

So clearly, S is non-blocking.

Finally, K controllable $\Rightarrow$ $pr(K)$ controllable $\Rightarrow$ $L(G||S)$ controllable
$\Rightarrow$ S Σ_u-enabling.

Example: $K = \{a^k b a^k \mid k \geq 0\}$ is controllable but not relative-closed

So desired non-blocking, non-marking, Σ_u-enabling supervisor does not exist

$K' = \{a^k b a^l \mid k \geq l \geq 0\}$

$pr(K') = pr(K)$ $\Rightarrow$ K' controllable (since K is controllable)

$pr(K') \cap L_m(G) = K'$ $\Rightarrow$ K' relative closed. So desired supervisor exists.

Tests for Existence of Supervisor

- $G = (X, \Sigma, \alpha, x_0, X_m)$ plant with m states

  $S = (Y, \Sigma, \beta, y_0, Y_m)$ trim acceptor for K with n states

  $\hookrightarrow L_m(S) = K$; $L(S) = pr(K)$

- Prefix closure: $[K = pr(K)] \Leftrightarrow [Re_S(y_0) \subseteq Y_m]$. ($O(n)$ test).

- Relative-closure: Construct $G \| S := (Z, \Sigma, \gamma, z_0, Z_m)$

  $\hookrightarrow K \subseteq L_m(G)$.

  $\hookrightarrow L_m(G\|S) = L_m(G) \cap K = K$; $L(G\|S) = L(G) \cap pr(K) = pr(K)$

  $[pr(K) \cap L_m(G) \subseteq K] \Leftrightarrow [X_m \times Y \subseteq X_m \times Y_m]$ ($O(mn)$ tests).

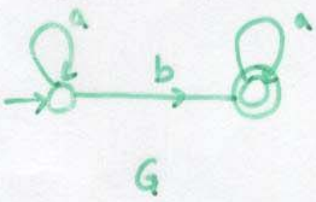
  $\cdot (x_0, y_0)$
  $\Big\} s \in L_m(G) \cap pr(K)$
  $\cdot (x,y) \in X_m \times Y$

- Controllability: Construct $G \| \bar{S} := (\bar{Z}, \Sigma, \bar{\gamma}, z_0, Z_m)$

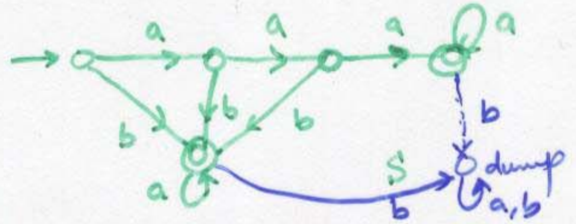
  $[pr(K)\Sigma_u \cap L(G) \subseteq pr(K)] \Leftrightarrow$

  $\forall (x,y) \in Z, \sigma \in \Sigma_u$: $\bar{\gamma}((x,y),\sigma)$ defined $\Rightarrow \bar{\gamma}((x,y),\sigma) \notin X \times \{y_D\}$

  Note: $L(G\|\bar{S}) = L(G) \cap \Sigma^* = L(G)$



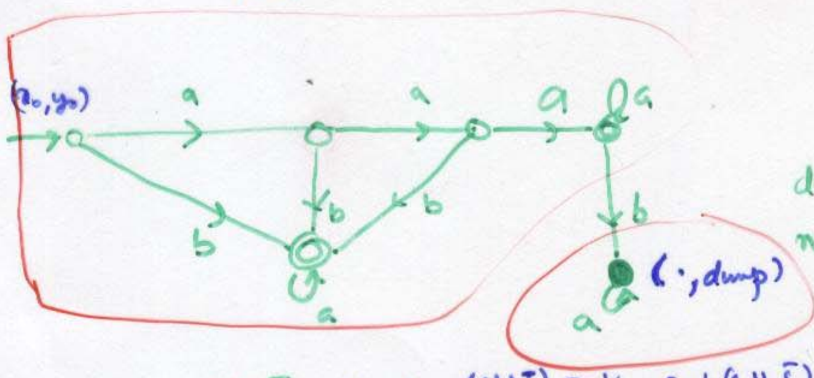
G



$L_m(\bar{S}) = L_m(S) = K$
$L(\bar{S}) = \Sigma^*$

block "b" after at most 2 initial "a"

$pr(K)$



dark state in $\bar{Z} - Z$: $L(G) - pr(K)$
non-dark state in Z : $pr(K)$

$L(G) - pr(K)$

$G\|\bar{S} \rightarrow L_m(G\|\bar{S}) = K$; $L(G\|\bar{S}) = L(G)$.

transition on uncontrollable event "b" from a state in Z to a state in $\bar{Z} - Z$