# FORMULA SHEET (Module 1)

**Motor model (Inputs: $\boldsymbol{V_m, T_d}$; Outputs: $\boldsymbol{\omega_m, I_m}$):** $V_m = R_m I_m + L_m \frac{dI_m}{dt} + V_{emf}; V_{emf} = k_m \omega_m (1 + \tau_m \omega_m)$

$J_{eq} \frac{d\omega_m}{dt} = T_m - T_d; T_m = k_t(I_m - I_f) = k_t[I_m - I_0 \operatorname{sgn}(\omega_m) - I_1 \omega_m - I_2 \omega_m^2]$; **Simplified:** $\omega_m = \frac{1}{J_{eq}s + \frac{k_m^2}{R_m}} \left(\frac{k_m}{R_m} V_m + T_d\right)$

**LSE:** $Y_N = A_N \theta \Rightarrow \boldsymbol{\hat{\theta}_N} = \left(\boldsymbol{A_N^T A_N}\right)^{-1} \boldsymbol{A_N^T Y_N}$

$$\sum_{j=0}^{n} a_j y(k-j) = \sum_{j=0}^{n} b_j u(k-j), \text{ and WLOG } a_0 = 1 \Rightarrow y(k) = \underbrace{[-y(k-1) \ldots - y(k-n) \;\; u(k) \ldots u(k-n)]}_{h^T(k)} \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_n \\ \vdots \\ b_n \end{bmatrix}}_{\theta}$$

$$\text{Collect } n \le k \le N \Rightarrow \underbrace{\begin{bmatrix} y(n) \\ \vdots \\ y(k) \\ \vdots \\ y(N) \end{bmatrix}_{N-n+1\times 1}}_{Y_N} = \underbrace{\begin{bmatrix} -y(n-1) \;\ldots\; -y(0) \;\; u(n) \;\ldots\; u(0) \\ \vdots \\ -y(k-1) \;\ldots\; -y(k-n) \;\; u(k) \;\ldots\; u(k-n) \\ \vdots \\ -y(N-1) \;\ldots\; -y(N-n) \;\; u(N) \;\ldots\; u(N-n) \end{bmatrix}_{N-n+1\times 2n+1}}_{H_N} \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_n \\ \vdots \\ b_n \end{bmatrix}_{2n+1\times 1}}_{\theta}$$

$\Rightarrow \boldsymbol{\hat{\theta}} = \left(\boldsymbol{H_N^T H_N}\right)^{-1} \boldsymbol{H_N^T Y_N}$

**LTI:** $y(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau \equiv h(t) * u(t) \leftrightarrow H(s)U(s) = U(s)H(s) \leftrightarrow u(t) * h(t) \equiv \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau$

**ZOH discretization:** $G(z) = \mathcal{Z}[H_{zoh}(s)G(s)] = \mathcal{Z}\left[\frac{1-e^{-sT}}{s} G(s)\right] = (1 - z^{-1})\mathcal{Z}\left[\frac{G(s)}{s}\right]$

**Bilinear discretization:** $s \approx \frac{2}{T}\frac{z-1}{z+1} \Leftrightarrow z \approx \frac{1+\frac{Ts}{2}}{1-\frac{Ts}{2}}$

**Poles, $\boldsymbol{p = \sigma \pm j\omega}$** $\Rightarrow$Time const, $\tau = \frac{1}{|\sigma|} = \frac{1}{|\text{Re}(p)|}$; Osc. freq, $\omega = \text{Im}(p)$; Nat. freq, $\omega_n = \sqrt{\sigma^2 + \omega^2}$; Damping ratio, $\zeta = -\sigma/\omega_n$

Note for a quadratic char. polynomial with poles at $p_1, p_2$: $\omega_n^2 = p_1 p_2$; $\zeta = \frac{-(p_1+p_2)}{2\sqrt{p_1 p_2}}$

**2nd-order TF,** $\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \Rightarrow$ Poles: $\begin{cases} -\omega_n(\zeta \pm (\sqrt{1-\zeta^2}); & (\zeta < 1) \\ -\omega_n; & (\zeta = 1) \\ -\omega_n(\zeta \pm \sqrt{\zeta^2 - 1}); & (\zeta > 1) \end{cases}$

Simple real poles, ie, over-damped $\Leftrightarrow \zeta > 1$: $1 - \left(\frac{\sqrt{\zeta^2-1}+\zeta}{2\sqrt{\zeta^2-1}}\right) e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} - \left(\frac{\sqrt{\zeta^2-1}-\zeta}{2\sqrt{\zeta^2-1}}\right) e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$

Repeated real poles, ie, critically-damped $\Leftrightarrow \zeta = 1$: $1 - e^{-\omega_n t}(1 + \omega_n t)$

Complex pair of poles, ie, under-damped $\Leftrightarrow \zeta < 1$: $1 - \left[\frac{1}{\sqrt{1-\zeta^2}}\right] e^{\sigma t}[\sin(\omega t + \cos^{-1}\zeta)]$; $\left(\sigma = -\zeta\omega_n; \omega = \sqrt{1-\zeta^2}\,\omega_n\right)$

**Peak overshoot** $e^{-\pi\zeta/\sqrt{1-\zeta^2}}$ @ half time-period, $\frac{\pi}{\omega} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$; **Rise time** (time to reach 1): $\frac{\pi - \cos^{-1}\zeta}{\omega}$

**Settling time** (3-5 time-const): $\frac{3}{\omega_n\zeta}$ (5%); $\frac{4}{\omega_n\zeta}$ (2%); $\frac{5}{\omega_n\zeta}$ (1%); **BW:** $\omega_n\sqrt{[(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}]}$;

**Bode peak** ($@\omega_n\sqrt{1-2\zeta^2}$)$= \frac{1}{2\zeta\sqrt{1-\zeta^2}}$ ($@\omega_n = \frac{1}{2\zeta}$); **Phase** ($@\omega_n\sqrt{1-2\zeta^2}$)$= -\tan^{-1}\left(\frac{\sqrt{1-2\zeta^2}}{\zeta}\right)$ ($@\omega_n = -90°$)

**Phase-margin,** $\left||\angle L(j\omega_g)| - \pi\right| = \tan^{-1}\left(\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1} - 2\zeta^2}}\right)$; **Gain-margin,** $\left|20\log_{10}\left(L(j\omega_p)\right)\right| = \infty$

| | $s$-domain $e_{ss} = \lim_{s\to 0} s\left[\frac{R(s)}{1+L(s)}\right]$ | $z$-domain $e_{ss} = \lim_{z\to 1}(z-1)\left[\frac{R(z)}{1+L(z)}\right]$ |
|---|---|---|
| $N = 0$ (step-input) $\Rightarrow R(s) = \frac{1}{s}$ or $R(z) = \frac{z}{z-1}$ | $\frac{1}{1 + \lim_{s\to 0} L(s)} \equiv \frac{1}{1 + K_0^s}$ | $\frac{1}{1 + \lim_{z\to 1} L(z)} \equiv \frac{1}{1 + K_0^z}$ |
| $N > 0 \Rightarrow R(s) = O\left(\frac{1}{s^{N+1}}\right)$ or $R(z) = O\left(\frac{T^N}{(z-1)^{N+1}}\right)$ | $\frac{1}{\lim_{s\to 0} s^N L(s)} \equiv \frac{1}{K_N^s}$ | $\frac{T^N}{\lim_{z\to 1}(z-1)^N L(z)} \equiv \frac{1}{K_N^z}$ |

Type $= N \Rightarrow e_{ss} = 1/(1 + K_N)$ and $1/K_N$ resp., for $O(t^0)$ and $O(t^N)$ inputs;

Type $> N \Rightarrow e_{ss} = 0 \wedge$ Type $< N \Rightarrow e_{ss} = \infty$ for $O(t^N)$ inputs;

**Char. eq:** $\chi(s) := \text{num}(1 + L(s)) = \text{num}(L(s)) + \text{den}(L(s))$

**ZN Tuning of PID:** $K_p = 0.6K_u, K_I = 1.2\frac{K_u}{T_u}; K_D = .075K_u T_u$

**BIBO Stable** $\Leftrightarrow \int_{-\infty}^{\infty}|h(t)|dt < \infty \Leftrightarrow$ All poles in LHP or if on imaginary-axis then non-repeating

**Internally Stable** $\Leftrightarrow$ All eigenvalues in LHP (asymp. stable) or if on imaginary-axis then grade-1

## FORMULA SHEET (Module 2)

**Lagrangian:** $L = KE - PE; \frac{\partial^2 L}{\partial t \partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = F_i$ **OR** $\tau_i$ (when $q_i$ is linear **OR** angular position)

**Linearization:** $\dot{x}(t) = f(x(t), u(t), t);\ y(t) = h(x(t), u(t), t)$ is at equilibrium at $x^*$ if $\exists u^*(t): f(x^*, u^*(t), t) = 0$. The linearized system matrices at the equilibrium are given by:

$$A(t) \equiv \left.\frac{\partial f}{\partial x}\right|_{x^*,u^*(t)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix},\ B(t) \equiv \left.\frac{\partial f}{\partial u}\right|_{x^*,u^*(t)} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix},$$

$$C(t) \equiv \left.\frac{\partial h}{\partial x}\right|_{x^*,u^*(t)} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial x_1} & \cdots & \frac{\partial h_p}{\partial x_n} \end{bmatrix},\ D(t) \equiv \left.\frac{\partial h}{\partial u}\right|_{x^*,u^*(t)} = \begin{bmatrix} \frac{\partial h_1}{\partial u_1} & \cdots & \frac{\partial h_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial u_1} & \cdots & \frac{\partial h_p}{\partial u_m} \end{bmatrix}.$$

**Transfer function:** $G(s) \equiv G_{sp}(s) + G(\infty)$, with $G_{sp}(s) = C(sI - A)^{-1}B$ and $G(\infty) = D$.

**State-space:** $G(s) = \frac{Y(s)}{U(s)} = \frac{b_n s^n + \ldots + b_0}{a_n s^n + \ldots + a_0} = \frac{b'_{n-1}s^{n-1} + \ldots + b'_0}{s^n + a'_{n-1}s^{n-1} + \ldots + a'_0} + \frac{b_n}{a_n}$, its **companion form** state-space realization is:

$$A = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & 1 \\ -a'_0 & -a'_1 & \cdots & \cdots & -a'_{n-1} \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},\ C = [b'_0\ b'_1\ \ldots\ b'_{n-1}],\ D = \left[\frac{b_n}{a_n}\right].$$

**State-eq solution:** $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$, where $e^{At} \leftrightarrow (sI - A)^{-1}$

**State change:** $z(t) = Px(t) \Rightarrow \hat{A} = P^{-1}AP;\ \hat{B} = P^{-1}B;\ \hat{C} = CP;\ \hat{D} = D.$

**Control input for steering initial state to final state:** $u(t) = -B^T e^{A^T(t_f - t)} W_c^{-1}(t_f)\left[e^{At_f}x(0) - x(t_f)\right]$, where $t_f > 0$ is the final time, and $W_c(t_f) = \int_0^{t_f} e^{A\tau}BB^T e^{A^T\tau}d\tau$.

**Controllability matrix:** $T = [B\ AB\ A^2B \cdots A^{n-1}B]$.

**Map to companion form:** Use similarity transform, $P = T\tilde{T}^{-1}$ (works for controllable system).

**State-feedback using companion form:** Given $\chi_A(s) = s^n + a_{n-1}s^{n-1} + \cdots a_0$, and desired $\chi_d(s) = s^n + \tilde{a}_{n-1}s^{n-1} + \cdots \tilde{a}_0$. Then, $\tilde{K} = [(\tilde{a}_0 - a_0)\ (\tilde{a}_1 - a_1) \ldots (\tilde{a}_{n-1} - a_{n-1})]$ and $K = \tilde{K}P^{-1} = \tilde{K}\,\tilde{T}\,T^{-1}$ are such that $\chi_{\tilde{A}-\tilde{B}\tilde{K}}(s) = \chi_d(s) = \chi_{A-BK}(s)$.

**Ackermann's formula** for state feedback control: $K = [0\ 0\ \ldots 1]T^{-1}\chi_d(A)$.

**Observability matrix:** $O = \begin{bmatrix} C \\ \vdots \\ CA^{n-1} \end{bmatrix}$.

**State Estimator/Observer:** $\left[\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \wedge \hat{y} = C\hat{x} + Du\right] \Rightarrow \dot{e} = (A - LC)e$, where $e = x - \hat{x}$.

**Synthesis of $L^T$** is like the synthesis of $K$ with $(A, B)$ replaced by $(A^T, C^T)$. Once $L^T$ is found, take the transpose to get $L$.

**Lyapunov function:** It's a positive function with negative rate: $\boldsymbol{V(x - x^*) \geq 0 \wedge \dot{V}(x - x^*) < 0 \wedge V(0) = \dot{V}(0) = 0}$, where $x^*$ is equilibrium.

For linear system, $\dot{x} = Ax$, it's equilibrium is $x^* = 0$, and one can choose $V(x - x^*) = \boldsymbol{V(x) = x^T P x}$. Then $\dot{V}(x) = \dot{x}^T Px + x^T P\dot{x} = x^T A^T Px + x^T PAx < 0$ iff $A^T P + PA < 0$ iff $\exists \boldsymbol{Q > 0}: \boldsymbol{A^T P + PA = -Q}$ and $V(x) = x^T Px \geq 0$ iff $P \geq 0$ (NOTE: $P \geq 0$ if all its eigenvalues are non-negative, and $Q > 0$ if all its eigenvalues are positive.)

**Sliding mode control:** For a sliding surface $s = 0$, consider energy function $V = \frac{1}{2}s^T s$
$\Rightarrow \dot{V} = s^T\dot{s} = s^T\frac{\partial s}{\partial x}\dot{x} = s^T\frac{\partial s}{\partial x}f(x, u)$. Choose $u$ so that $\operatorname{sgn}(f(x, u)) = -\operatorname{sgn}\left(s^T\frac{\partial s}{\partial x}\right) \Rightarrow \dot{V} \leq 0$.

## FORMULA SHEET (Module 3)

**Rotation Matrix is Unitary,** ie, $R^{-1} = R^T \wedge det(R) = 1$

**2D rotation; rotation+translation:** $R_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}; T\left(\psi; \begin{bmatrix} x_t \\ z_t \end{bmatrix}\right) = \begin{bmatrix} cos\,\psi & -sin\,\psi & x_t \\ sin\,\psi & cos\,\psi & y_t \\ 0 & 0 & 1 \end{bmatrix}$

**3D rotation:** $R(\phi, \theta, \psi) = R_z(\psi)R_y(\theta)R_x(\phi) = \begin{bmatrix} \cos\theta\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi \\ \cos\theta\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix}$

**Quadcopter model:**

$$\left.\begin{aligned} m\ddot{x} &= (cos\,\phi\,sin\,\theta\,cos\,\psi + sin\,\phi\,sin\,\psi)u_1 \\ m\ddot{y} &= (cos\,\phi\,sin\,\theta\,sin\,\psi - sin\,\phi\,cos\,\psi)u_1 \\ m\ddot{z} &= (cos\,\phi\,cos\,\theta)u_1 - mg \\ I_{xx}\ddot{\phi} - (I_{yy} - I_{zz})\dot{\theta}\dot{\psi} + I_r\dot{\theta}\omega &= \ell u_2 \\ I_{yy}\ddot{\theta} - (I_{zz} - I_{xx})\dot{\psi}\dot{\phi} - I_r\dot{\phi}\omega &= \ell u_3 \\ I_{zz}\ddot{\psi} - (I_{xx} - I_{yy})\dot{\phi}\dot{\theta} &= \ell u_4 \end{aligned}\right\},\ \text{where:} \begin{cases} u_1 = K_T(\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2) \\ u_2 = K_T(\omega_2^2 - \omega_4^2) \\ u_3 = K_T(\omega_1^2 - \omega_3^2) \\ u_4 = K_D(\omega_2^2 + \omega_4^2 - \omega_1^2 - \omega_3^2) \\ \omega = \omega_2 + \omega_4 - \omega_1 - \omega_3 \end{cases}$$

**Quaternion,** $q = a + ib + jc + kd$, with $i^2 = j^2 = k^2 = ijk = -1 \Rightarrow (ij = k, jk = i, ki = j) \wedge (ji = -k, kj = -i, ik = -j)$

**Rotate $\boldsymbol{p}$ around $\boldsymbol{u}$ by $\boldsymbol{\delta}$:** $p' = rpr^*$, with $r = \cos\frac{\delta}{2} + \left(u_x\sin\frac{\delta}{2}\right)i + \left(u_y\sin\frac{\delta}{2}\right)j + \left(u_z\sin\frac{\delta}{2}\right)k$, where $\boldsymbol{u_x^2 + u_y^2 + u_z^2 = 1}$.

**Same rotation using** $R_u(\delta) = \begin{bmatrix} \cos\delta + u_x^2(1-\cos\delta) & u_x u_y(1-\cos\delta) - u_z\sin\delta & u_z u_x(1-\cos\delta) + u_y \sin\delta \\ u_x u_y(1-\cos\delta) + u_z\sin\delta & \cos\delta + u_y^2(1-\cos\delta) & u_y u_z(1-\cos\delta) - u_x\sin\delta \\ u_z u_x(1-\cos\delta) - u_y\sin\delta & u_y u_z(1-\cos\delta) + u_x\sin\delta & \cos\delta + u_z^2(1-\cos\delta) \end{bmatrix}$

**Attitude & altitude PID:** $\left.\begin{aligned} u_1 &:= K_{P_z}\bar{z} + K_{I_z}\int \bar{z} + K_{D_z}\frac{d\bar{z}}{dt} \\ u_2 &:= K_{P_\phi}\bar{\phi} + K_{I_\phi}\int \bar{\phi} + K_{D_\phi}\frac{d\bar{\phi}}{dt} \\ u_3 &:= K_{P_\theta}\bar{\theta} + K_{I_\theta}\int \bar{\theta} + K_{D_\theta}\frac{d\bar{\theta}}{dt} \\ u_4 &:= K_{P_\psi}\bar{\psi} + K_{I_\psi}\int \bar{\psi} + K_{D_\psi}\frac{d\bar{\psi}}{dt} \end{aligned}\right\}$, where $\bar{z} := z - z_d; \bar{\phi} := \phi - \phi_d; \bar{\theta} := \theta - \theta_d; \bar{\psi} := \psi - \psi_d$

**Position & yaw PID = Same as above PID with:** $\phi_d = \tan^{-1}\left(\frac{\bar{x}\sin\psi - \bar{y}\cos\psi}{\sqrt{(\bar{x}\cos\psi + \bar{y}\sin\psi)^2 + \bar{z}^2}}\right) \wedge \theta_d = \tan^{-1}\left(\frac{\bar{x}\cos\psi + \bar{y}\sin\psi}{\bar{z}}\right)$

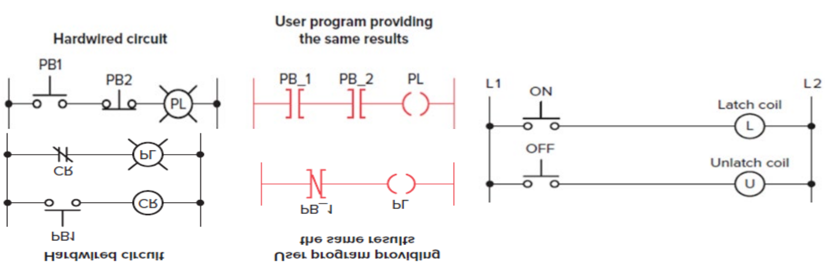
**Kalman Filter:**

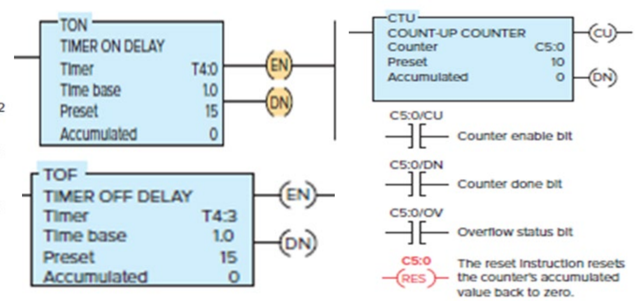
$$\left.\begin{aligned} R_{\underline{e}}(k+1) &= AR_e(k)A^T + R_\omega(k) \\ G(k+1) &= R_{\underline{e}}(k+1)C^T\left[CR_{\underline{e}}(k+1)C^T + R_v(k+1)\right]^{-1} \\ R_e(k+1) &= (I - G(k+1)C)R_{\underline{e}}(k+1) \end{aligned}\right\} \wedge \left\{\begin{aligned} \bar{x}(k+1) &= A\hat{x}(k) + Bu(k) \\ \bar{y}(k+1) &= C\bar{x}(k+1) + Du(k+1) \\ \hat{x}(k+1) &= \bar{x}(k+1) + G(k+1)[y(k+1) - \bar{y}(k+1)] \end{aligned}\right.$$

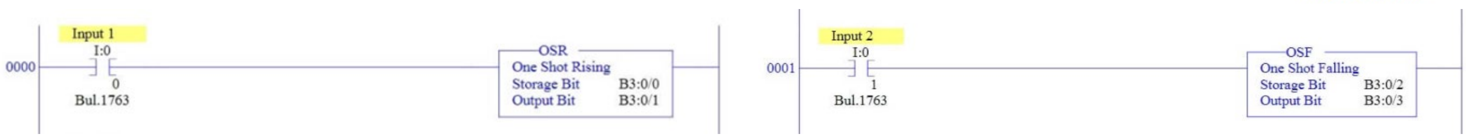
## FORMULA SHEET (Module 4)

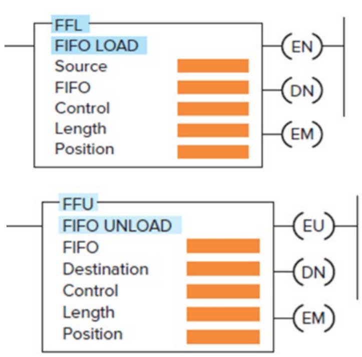
**Inputs:** XIC, XIO

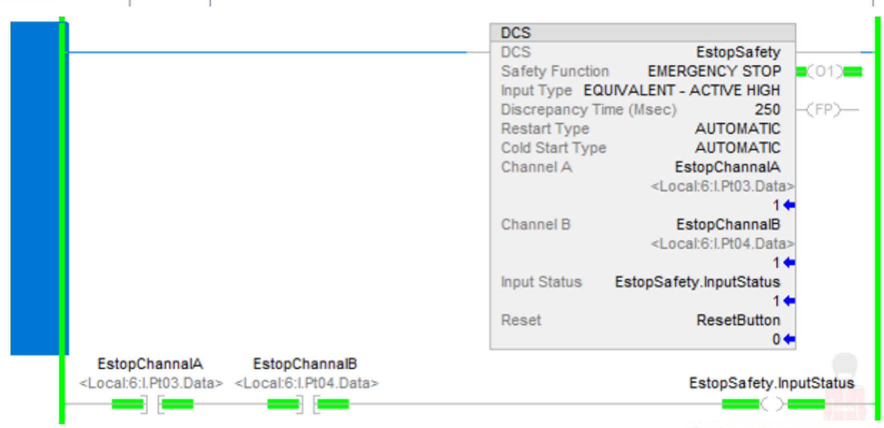
**Outputs:** OTE, Latch/Unlatch, Timer: On-delay vs Off Delay, Counter, One-shot Rising vs Falling, FIFO, DCS

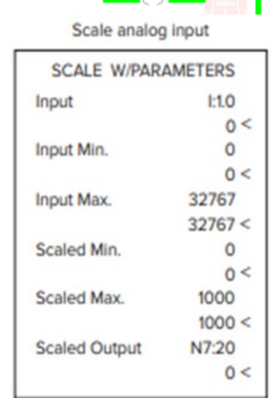



**Inputs:** Compare with Source A and Source B (EQU, NEQ, GRT, GEQ, LES, LEQ)

**Outputs:** Ops with Sources/inputs & Destination/Output (MOV, ADD, SUB, MUL, DIV, MOD, SQR, SCL, SCP)

**Program Controls:** Output/Input pairs (JMP/LBL, JSR/RET)