Section 8.3
Representing Relations

Connection Matrices

Let R be a relation from

$$A = \{a_1, a_2, \ldots, a_m\}$$

to

$$B = \{b_1, b_2, \ldots, b_n\}.$$

Definition: An *m* x*n connection matrix M* for R is defined by

$$M_{ij} = 1 \text{ if } <a_i, b_j> \text{ is in R,}$$
$$= 0 \text{ otherwise.}$$

Example:

We assume the rows are labeled with the elements of A and the columns are labeled with the elements of B.

Let

$$A = \{a, b, c\}$$

$$B = \{e, f, g, h\}$$

$$R = \{<a,e>, <c, g>\}$$

Then the connection matrix M for R is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Note: the order of the elements of A and B matters.

Theorem: Let R be a binary relation on a set A and let M be its connection matrix. Then

- R is reflexive iff $M_{ii} = 1$ for all i.
- R is symmetric iff M is a symmetric matrix: $M = M^T$
- R is antisymetric if $M_{ij} = 0$ or $M_{ji} = 0$ for all $i \neq j$.

Combining Connection Matrices

Definition: the *join* of two matrices M_1, M_2, denoted $M_1 \quad M_2$, is the component wise boolean ‘or’ of the two matrices.

Fact: If M_1 is the connection matrix for R_1 and M_2 is the connection matrix for R_2 then the join of M_1 and M_2, $M_1 \quad M_2$ is the connection matrix for $R_1 \quad R_2$.

Definition: the *meet* of two matrices M_1, M_2, denoted $M_1 \quad M_2$ is the componentwise boolean 'and' of the two matrices.

Fact: If M_1 is the connection matrix for R_1 and M_2 is the connection matrix for R_2 then the meet of M_1 and M_2, $M_1 \quad M_2$ is the connection matrix for $R_1 \cap R_2$.

Obvious questions:

Given the connection matrix for two relations, how does one find the connection matrix for

- The complement?
- The relative complement?
- The symmetric difference?

The Composition

Definition: Let

M_1 be the connection matrix for R_1 and

M_2 be the connection matrix for R_2.

The *boolean product* of two connection matrices M_1 and M_2, denoted $M_1 \quad M_2$, is the connection matrix for the composition of R_2 with R_1, $R_2 \circ R_1$.

$$(M_1 \quad M_2)_{ij} = \underset{k=1}{\overset{n}{}}[(M_1)_{ik} \quad (M_2)_{kj}]$$

Why?

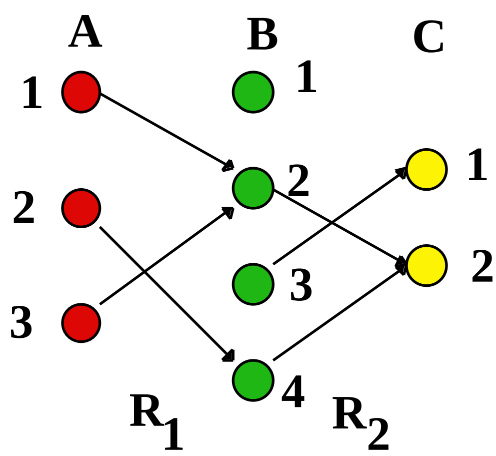
In order for there to be an arc <x, z> in the composition then there must be and arc <x, y> in R_1 and an arc <y, z> in R_2 *for some y* !

The Boolean product checkes all possible y's. If at least one such path exists, that is sufficient.

Note: the matrices M_1 and M_2 must be *conformable:* the number of columns of M_1 must equal the number of rows of M_2.

If M_1 is mxn and M_2 is nxp then $M_1 \quad M_2$ is mxp.

Example:



$$M_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$M_1 \quad M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$(M_1 \quad M_2)_{12} = [(M_1)_{11} \quad (M_2)_{12}] \quad [(M_1)_{12} \quad (M_2)_{22}]$$
$$[(M_1)_{13} \quad (M_2)_{32}] \quad [(M_1)_{14} \quad (M_2)_{42}]$$

$$= [0 \quad 0] \quad [1 \quad 1] \quad [0 \quad 0] \quad [0 \quad 1] = 1$$

Note:

• there is an arc in R_1 from node 1 in A to node 2 in B

• there is an arc in R_2 from node 2 in B to node 2 in C.

• Hence there is an arc in $R_2 \circ R_1$ from node 1 in A to node 2 in C.

A useful result:

$$M_{R^n} = M_R^n$$

Digraphs

(see section 8.1)

Given the digraphs for R_1 and R_2, find the digraphs for

- $R_2 \quad R_1$
- $R_2 \cap R_1$
- $R_2 - R_1$

- $R_2 \quad R_1$
- $\overline{R}_1$
