Section 1.2
Propositional Equivalences

A *tautology* is a proposition which is always true.

Classic Example: $P \quad \neg P$

A *contradiction* is a proposition which is always false.

Classic Example: $P \quad \neg P$

A *contingency* is a proposition which neither a tautology nor a contradiction.

Example: $(P \quad Q) \rightarrow \neg R$

Two propositions P and Q are *logically equivalent* if $P \leftrightarrow Q$ is a tautology. We write

$$P \Leftrightarrow Q$$

Example: $(P \rightarrow Q) \quad (Q \rightarrow P) \Leftrightarrow (P \leftrightarrow Q)$

Proof:

The left side and the right side must have the same truth values <u>independent</u> of the truth value of the component propositions.

To show a proposition is not a tautology: use an *abbreviated* truth table

- try to find a *counter example* or to *disprove* the assertion.

- search for a case where the proposition is false

Case 1: Try left side false, right side true

Left side false: only one of $P \rightarrow Q$ or $Q \rightarrow P$ need be false.

1a. Assume $P \rightarrow Q$ = F.
Then P = T , Q = F. But then right side $P \leftrightarrow Q$ = F. Oops, wrong guess.

1b. Try $Q \rightarrow P$ = F. Then Q = T, P = F. Then $P \leftrightarrow Q$ = F. Another wrong guess.

Case 2. Try left side true, right side false

If right side is false, P and Q cannot have the same truth value.

2a. Assume P =T, Q = F.
Then $P \rightarrow Q$ = F and the conjunction must be false so the left side cannot be true in this case. Another wrong guess.

2b. Assume Q = T, P = F.
Again the left side cannot be true.

We have exhausted all possibilities and not found a counterexample. The two propositions must be logically equivalent.

Note: Because of this equivalence, *if and only if* or *iff* is also stated as *is a necessary and sufficient condition for*.

Some famous logical equivalences:

Logical Equivalences

$P \quad T \Leftrightarrow P$ $P \quad F \Leftrightarrow P$	Identity
$P \quad T \Leftrightarrow T$ $P \quad F \Leftrightarrow F$	Domination
$P \quad P \Leftrightarrow P$ $P \quad P \Leftrightarrow P$	Idempotency
$\neg(\neg P)) \Leftrightarrow P$	Double negation
$P \quad Q \Leftrightarrow Q \quad P$ $P \quad Q \Leftrightarrow Q \quad P$	Commutativity
$(P \quad Q) \quad R \Leftrightarrow P \quad (Q \quad R)$ $(P \quad Q) \quad R \Leftrightarrow P \quad (Q \quad R)$	Associativity

Distributivity

$P \quad (Q \quad R) \Leftrightarrow$

$(P \quad Q) \quad (P \quad R)$

$P \quad (Q \quad R) \Leftrightarrow$

$(P \quad Q) \quad (P \quad R)$

DeMorgan's laws

$\neg(P \quad Q) \Leftrightarrow \neg P \quad \neg Q$

$\neg(P \quad Q) \Leftrightarrow \neg P \quad \neg Q$

Implication

$P \rightarrow Q \Leftrightarrow \neg P \quad Q$

Tautology

$P \quad \neg P \Leftrightarrow T$

Contradiction

$P \quad \neg P \Leftrightarrow F$

$P \quad T \Leftrightarrow P$

$P \quad F \Leftrightarrow P$

Equivalence

$(P \rightarrow Q) \quad (Q \rightarrow P) \Leftrightarrow$

$(P \leftrightarrow Q)$

Absurdity

$(P \rightarrow Q) \quad (P \rightarrow \neg Q) \Leftrightarrow$

$\neg P$

Contrapositive

$(P \rightarrow Q) \Leftrightarrow (\neg Q \rightarrow \neg P)$

Absorption

$P \quad (P \quad Q) \Leftrightarrow P$

$P \quad (P \quad Q) \Leftrightarrow P$

Exportation

$(P \quad Q) \rightarrow R \Leftrightarrow$

$P \rightarrow (Q \rightarrow R)$

Note: equivalent expressions can always be substituted for each other in a more complex expression - useful for simplification.

Normal **or** ***Canonical*** **Forms**

Unique representations of a proposition

Examples:

Construct a <u>simple</u> proposition of two variables which is true only when

- P is true and Q is false:

$$P \quad \neg Q$$

- P is true and Q is true:

$$P \quad Q$$

- P is true and Q is false or P is true and Q is true:

$$(P \quad \neg Q) \quad (P \quad Q)$$

A disjunction of conjunctions where

- every variable or its negation is represented once in each conjunction (a *minterm)*

- each minterms appears only once

Disjunctive Normal Form (DNF)

Important in switching theory, simplification in the design of circuits.

Method: To find the minterms of the DNF.

- Use the rows of the truth table where the proposition is 1 or True
- If a zero appears under a variable, use the negation of the propositional variable in the minterm
- If a one appears, use the propositional variable.

Example:

Find the DNF of $(P \quad Q) \rightarrow \neg R$

P	Q	R	$(P \quad Q) \rightarrow \neg R$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

There are 5 cases where the proposition is true, hence 5 minterms. Rows 1,2,3, 5 and 7 produce the following disjunction of minterms:

$$(P \quad Q) \rightarrow \neg R$$

$$\Leftrightarrow (\neg P \quad \neg Q \quad \neg R) \quad (\neg P \quad \neg Q \quad R) \quad (\neg P \quad Q \quad \neg R)$$
$$(P \quad \neg Q \quad \neg R) \quad (P \quad Q \quad \neg R)$$

Note that you get a *Conjunctive Normal Form* (CNF) if you negate a DNF and use DeMorgan's Laws.
