Modified-CS: modifying Compressive Sensing for problems with partially known support

Namrata Vaswani and Wei Lu (ISU)

Department of Electrical and Computer Engineering
Iowa State University
Web: http://www.ece.iastate.edu/~namrata/research/SequentialCS.html

The Problem

- Reconstruct an m-length sparse vector, x, from an n-length measurement vector, $y := Ax$, when $n < m$
  - *use partial knowledge of the support of x to reduce the n required for exact reconstruction*
- Support of x is $N = T \cup \Delta \setminus \Delta_e$
  - T: "known" part of the support
  - Δ: "unknown" part of the support, Δ is disjoint with T
  - $\Delta_e \subseteq T$: error in the known part
- Measurement matrix, A, satisfies the S-RIP, $S = |T| + 2|\Delta|$

Application - 1: single signal/image

- T known from prior knowledge,
- e.g. in a natural image (often wavelet-sparse) with a small black background, most approximation (lowest subband) coeff's will be nonzero
- Set $T = \{$indices of all approximation coeff's$\}$, then
  - $\Delta_e = \{$indices of approximation coeff's which are zero$\}$
  - $\Delta = \{$indices of scaling coeff's which are nonzero$\}$

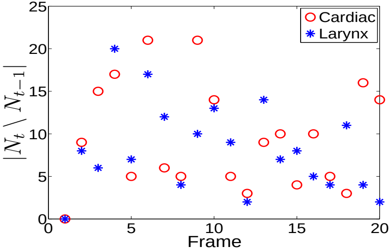
Application - 2: time sequence of signals/images

- Recursively reconstruct a time sequence of sparse vectors, x_t, with support, N_t, from measurement vectors, $y_t := Ax_t$
  - "recursively": use only $\hat{x}_{t-1}$ and y_t to reconstruct x_t
- Applications: real-time dynamic MRI, single-pixel video, ...
  - *use: the sparsity pattern of the signal sequence changes slowly*
- Set $T = \hat{N}_{t-1}$ (support estimate from $t-1$)

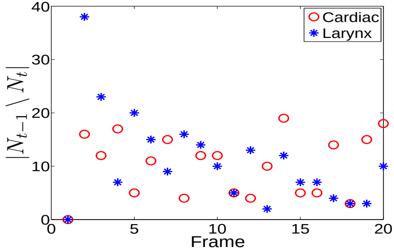
Application - 2: time sequence of signals/images

- Recursively reconstruct a time sequence of sparse vectors, x_t, with support, N_t, from measurement vectors, $y_t := Ax_t$
  - "recursively": use only $\hat{x}_{t-1}$ and y_t to reconstruct x_t
- Applications: real-time dynamic MRI, single-pixel video, ...
  - *use: the sparsity pattern of the signal sequence changes slowly*
- Set $T = \hat{N}_{t-1}$ (support estimate from $t-1$)
  - if $\hat{N}_{t-1} = N_{t-1}$ (exact recon), $\Delta = N_t \setminus N_{t-1}$, $\Delta_e = N_{t-1} \setminus N_t$
  - slow changes in sparsity pattern $\iff |\Delta|, |\Delta_e| << |N_t| \approx |T|$

Example: slow support change of medical image seq's



(a) additions: $|N_t \setminus N_{t-1}|$



(b) deletions: $|N_{t-1} \setminus N_t|$

- N_t: 99%-energy support of 2D-DWT of image
- **Maximum size of addition or deletion less than** $|N_t|/50$ for both
  - heart: $|N_t| \approx 1400 - 1500$, $m = 4096$
  - larynx: $|N_t| \approx 4400 - 4600$, $m = 65536$

Outline

Notation, Recap of Compressive Sensing [Donoho'05,Candes,Romberg,Tao'05]

- Notation:
  - A_T: sub-matrix containing columns of A with indices in set T
  - β_T: sub-vector containing elements of β with indices in set T
  - $T^c = [1:m] \setminus T$: complement of set T
  - $||A||$: spectral matrix norm (induced 2-norm)
  - A': denotes the transpose of matrix A

- **Compressive Sensing:** Reconstructs a sparse signal, x, with support, N, from $y := Ax$ by solving

$$\min_{\beta} ||\beta||_1 \ s.t. \ y = A\beta$$

  - Exact reconstruction will occur if $\delta_{2|N|} + \theta_{|N|,2|N|} < 1$

Define δ_S, $\theta_{S,S'}$ [Candes,Romberg,Tao'05]

- Restricted isometry constant, δ_S: smallest real number s.t.

$$(1-\delta_S)||c||_2^2 \leq ||A_T c||_2^2 \leq (1+\delta_S)||c||_2^2$$

  $\forall$ subsets T with $|T| \leq S$ and for all c

  - easy to see: $||(A_T{}'A_T)^{-1}|| \leq 1/(1-\delta_{|T|})$

- Restricted orthogonality constant, $\theta_{S,S'}$: smallest real no. s.t.

$$|c_1{}'A_{T_1}{}'A_{T_2}c_2| \leq \theta_{S,S'}||c_1||_2||c_2||_2$$

  $\forall$ disjoint sets T_1, T_2 with $|T_1| \leq S$, $|T_2| \leq S'$ and $\forall$ c_1, c_2

  - easy to see: $||A_{T_1}{}'A_{T_2}|| \leq \theta_{|T_1|,|T_2|}$

Modified-CS [Vaswani,Lu, ISIT'09]

- Our problem: reconstruct x with support $N = T \cup \Delta \setminus \Delta_e$ from $y := Ax$ when T is known
- First consider the case: $|\Delta_e| = 0$, i.e. $N = T \cup \Delta$
  - among all solutions of $y = A\beta$, find the β whose support contains the smallest number of new additions to T

Modified-CS [Vaswani,Lu, ISIT'09]

- Our problem: reconstruct x with support $N = T \cup \Delta \setminus \Delta_e$ from $y := Ax$ when T is known
- First consider the case: $|\Delta_e| = 0$, i.e. $N = T \cup \Delta$
  - among all solutions of $y = A\beta$, find the β whose support contains the smallest number of new additions to T

$$\min_{\beta} ||(\beta)_{T^c}||_0 \ s.t. \ y = A\beta$$

  - x is its unique solution if $\delta_S < 1$ for $S = |T| + 2|\Delta|$ (S-RIP)

Modified-CS [Vaswani,Lu, ISIT'09]

- Our problem: reconstruct x with support $N = T \cup \Delta \setminus \Delta_e$ from $y := Ax$ when T is known
- First consider the case: $|\Delta_e| = 0$, i.e. $N = T \cup \Delta$
  - among all solutions of $y = A\beta$, find the β whose support contains the smallest number of new additions to T

$$\min_{\beta} ||(\beta)_{T^c}||_0 \; s.t. \; y = A\beta$$

  - x is its unique solution if $\delta_S < 1$ for $S = |T| + 2|\Delta|$ (S-RIP)
- *The above also holds when* $|\Delta_e| \neq 0$, *i.e.* $N = T \cup \Delta \setminus \Delta_e$

Modified-CS [Vaswani,Lu, ISIT'09]

- Our problem: reconstruct x with support $N = T \cup \Delta \setminus \Delta_e$ from $y := Ax$ when T is known
- First consider the case: $|\Delta_e| = 0$, i.e. $N = T \cup \Delta$
  - among all solutions of $y = A\beta$, find the β whose support contains the smallest number of new additions to T

$$\min_\beta ||(\beta)_{T^c}||_0 \ s.t. \ y = A\beta$$

  - x is its unique solution if $\delta_S < 1$ for $S = |T| + 2|\Delta|$ (S-RIP)
- *The above also holds when* $|\Delta_e| \neq 0$, *i.e.* $N = T \cup \Delta \setminus \Delta_e$
- Replace ℓ_0 norm by ℓ_1 norm: get a convex problem

$$\min_\beta ||(\beta)_{T^c}||_1 \ s.t. \ y = A\beta \quad \textbf{(modified-CS)}$$

Exact reconstruction using ℓ_0 modified-CS [Vaswani,Lu, ISIT'09]

$$\min_{\beta} ||\beta_{T^c}||_0 \ s.t. \ y = A\beta \quad (\ell_0 \text{ mod-CS})$$

- (ℓ_0 mod-CS) achieves exact reconstruction if
  - $\delta_{|T|+2|\Delta|} = \delta_{|N|+|\Delta_e|+|\Delta|} < 1$
  - if $|\Delta| = |\Delta_e| = |N|/50$, this becomes $\delta_{1.04|N|} < 1$
- Compare with CS
  - (ℓ_0 CS) needs $\delta_{2|N|} < 1$

recall: $T = N \cup \Delta_e \setminus \Delta$, T: known part of support, Δ: unknown part, Δ_e: error in known part

Exact reconstruction using modified-CS [Vaswani,Lu, ISIT'09]

$$\min_{\beta} ||\beta_{T^c}||_1 \ s.t. \ y = A\beta \quad \text{(mod-CS)}$$

Theorem

x is the unique minimizer of (mod-CS) if $\delta_{|T|+|\Delta|} < 1$ *and*

$$\theta_{|\Delta|,|\Delta|} + \delta_{2|\Delta|} + \theta_{|\Delta|,2|\Delta|} + \delta_{|T|} + \theta^2_{|\Delta|,|T|} + 2\theta^2_{2|\Delta|,|T|} < 1$$

Corollary (simplified condition)

x is the unique minimizer of (mod-CS) if $|\Delta| \leq |T|$ *and*

$$\delta_{|T|+2|\Delta|} < 1/5$$

recall: $T = N \cup \Delta_e \setminus \Delta$, T: known part of support, Δ: unknown part, Δ_e: error in known part

Comparing modified-CS with CS

- Compare sufficient conditions for exact recon.
  - *Modified-CS:* $\delta_{|T|+|\Delta|} < 1$ and

    $$Mcond = (\delta_{2|\Delta|} + \theta_{|\Delta|,|\Delta|} + \theta_{|\Delta|,2|\Delta|}) + (\delta_{|T|} + \theta^2_{|\Delta|,|T|} + 2\theta^2_{2|\Delta|,|T|}) < 1$$

  - *CS* [Decoding by LP, Candes,Tao'05]:

    $$Ccond = \delta_{2|N|} + \theta_{|N|,|N|} + \theta_{|N|,2|N|} < 1$$

  - If $|\Delta| \approx |\Delta_e| << |N|$ (typical for medical image seq's),

    $$Mcond << Ccond$$

recall: n is the number of measurements, T: known part of support, Δ: unknown part, Δ_e: error in known part

Comparing modified-CS with CS

- Compare sufficient conditions for exact recon.
  - *Modified-CS:* $\delta_{|T|+|\Delta|} < 1$ and
    $$Mcond = (\delta_{2|\Delta|} + \theta_{|\Delta|,|\Delta|} + \theta_{|\Delta|,2|\Delta|}) + (\delta_{|T|} + \theta^2_{|\Delta|,|T|} + 2\theta^2_{2|\Delta|,|T|}) < 1$$
  - *CS* [Decoding by LP, Candes,Tao'05]:
    $$Ccond = \delta_{2|N|} + \theta_{|N|,|N|} + \theta_{|N|,2|N|} < 1$$
  - If $|\Delta| \approx |\Delta_e| << |N|$ (typical for medical image seq's),
    $$Mcond << Ccond$$
- $Mcond < 1$ can hold even with $n < 2|N|$
- In simulations with $n = 1.6|N|$:
  - *Modified-CS worked w.p. 0.99* if $|\Delta| = |\Delta_e| = |N|/50$
  - *CS worked w.p. 0*

recall: n is the number of measurements, T: known part of support, Δ: unknown part, Δ_e: error in known part

Proof Idea: Lemma 1

- The idea of the proof is motivated by the proof of the exact reconstruction result for CS [Decoding by LP, Candes,Tao'05]

Lemma

x is the unique minimizer of (mod-CS) if $\delta_{|T|+|\Delta|} < 1$ and if we can find a vector w satisfying

- $A_T{}'w = 0$
- $A_\Delta{}'w = sgn(x_\Delta)$
- $||A_{(T \cup \Delta)^c}{}'w||_\infty < 1$

Proof Idea: Lemma 2

Lemma

Let c be a vector supported on a set T_d, that is disjoint with T, of size $|T_d| \leq S$, and let $\delta_S + \delta_{|T|} + \theta^2_{|T|,S} < 1$.
Then there exists a vector $\tilde{w}$ and an exceptional set, E, disjoint with $T \cup T_d$, of size $|E| < S'$ s.t. $||\tilde{w}||_2 \leq K_{|T|}(S)||c||_2$,

$$\begin{aligned} {A_T}'\tilde{w} &= 0, \\ {A_{T_d}}'\tilde{w} &= c, \\ ||{A_E}'\tilde{w}||_2 &\leq a_{|T|}(S,S')||c||_2, \\ ||{A_{(T\cup T_d \cup E)^c}}'\tilde{w}||_\infty &\leq \frac{a_{|T|}(S,S')}{\sqrt{S'}}||c||_2, \ \textit{where} \\ a_{|T|}(S,S') &:= \frac{\theta_{S',S} + \frac{\theta_{S',|T|}\ \theta_{S,|T|}}{1-\delta_{|T|}}}{1 - \delta_S - \frac{\theta^2_{S,|T|}}{1-\delta_{|T|}}} \end{aligned}$$

Proof idea: Proving the result

- Apply Lemma 2 iteratively to find a w needed by Lemma 1
  - at iteration $k = 0$, apply it with $S = S' = |\Delta|$: get $\tilde{w}_1$
  - at iteration $k > 0$, apply it with $S = 2|\Delta|$, $S' = |\Delta|$: get $\tilde{w}_{k+1}$
  - define $w = \sum_k (-1)^{k-1} \tilde{w}_k$
  - can show that w satisfies $A_T{}'w = 0$, $A_\Delta{}'w = \text{sgn}(x_\Delta)$, and

$$||A_{(T\cup\Delta)^c}{}'w||_\infty < a_{|T|}(2|\Delta|, |\Delta|) + a_{|T|}(|\Delta|, |\Delta|)$$

- Thus if $a_{|T|}(2|\Delta|, |\Delta|) + a_{|T|}(|\Delta|, |\Delta|) < 1$, Lemma 1 applies

Simulation results: Probability of exact reconstruction

- fixed $m = 256$, $|N| = 0.1m$ (typical 99%-support size)
- used a random Gaussian A
- varied n (number of measurements), $|\Delta|$ and $|\Delta_e|$
  - for each choice, averaged over N, $(x)_N$, Δ, Δ_e

recall: n is number of measurements, Δ: unknown part of support, Δ_e: error in known part

Simulation results: Probability of exact reconstruction

- fixed $m = 256$, $|N| = 0.1m$ (typical 99%-support size)
- used a random Gaussian A
- varied n (number of measurements), $|\Delta|$ and $|\Delta_e|$
  - for each choice, averaged over N, $(x)_N$, Δ, Δ_e

- For $n = 1.6|N|$,
  - CS works 0% times
  - Mod-CS works $\geq$ 99% times if $|\Delta| \leq 0.04|N|$, $|\Delta_e| \leq 0.08|N|$

recall: n is number of measurements, Δ: unknown part of support, Δ_e: error in known part

Simulation results: Probability of exact reconstruction

- fixed $m = 256$, $|N| = 0.1m$ (typical 99%-support size)
- used a random Gaussian A
- varied n (number of measurements), $|\Delta|$ and $|\Delta_e|$
  - for each choice, averaged over N, $(x)_N$, Δ, Δ_e

- For $n = 1.6|N|$,
  - CS works 0% times
  - Mod-CS works $\geq$ 99% times if $|\Delta| \leq 0.04|N|$, $|\Delta_e| \leq 0.08|N|$

- For $n = 2.5|N|$
  - CS works 0.2% times
  - Mod-CS works $\geq$ 99% times if $|\Delta| \leq 0.20|N|$, $|\Delta_e| \leq 0.24|N|$

recall: n is number of measurements, Δ: unknown part of support, Δ_e: error in known part

Simulation results: Probability of exact reconstruction

- fixed $m = 256$, $|N| = 0.1m$ (typical 99%-support size)
- used a random Gaussian A
- varied n (number of measurements), $|\Delta|$ and $|\Delta_e|$
  - for each choice, averaged over N, $(x)_N$, Δ, Δ_e

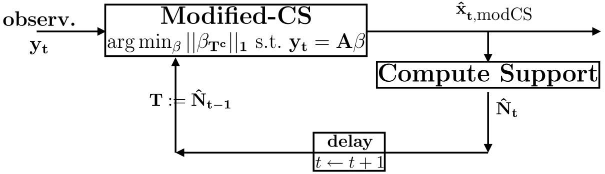
- For $n = 1.6|N|$,
  - CS works 0% times
  - Mod-CS works $\geq$ 99% times if $|\Delta| \leq 0.04|N|$, $|\Delta_e| \leq 0.08|N|$

- For $n = 2.5|N|$
  - CS works 0.2% times
  - Mod-CS works $\geq$ 99% times if $|\Delta| \leq 0.20|N|$, $|\Delta_e| \leq 0.24|N|$

- $n = 4|N|$: CS works 98% of times

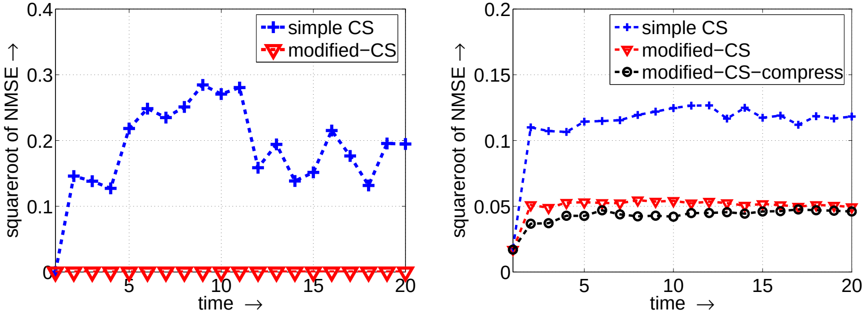
recall: n is number of measurements, Δ: unknown part of support, Δ_e: error in known part

Modified-CS for a time sequence of signals



- Support computed as $\hat{N}_t = \{i \in [1:m] : (\hat{x}_t)_i^2 > \alpha\}$

Simulated MRI: Reconstructing a cardiac image sequence



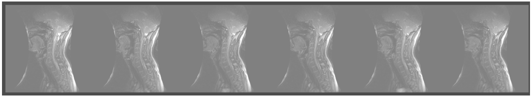
(c) $n = 0.16m$, Sparsified sequence (d) $n = 0.19m$, True sequence

- sparsity basis: 2-level 2D-DWT, Daubechies-4 wavelet
- $m = 1024$ (32x32 image), $|N_t| \approx 107 \approx 0.1m$

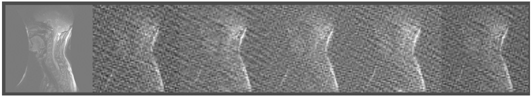
recall: n is number of measurements

Simulated MRI: Reconstructing a larynx image sequence

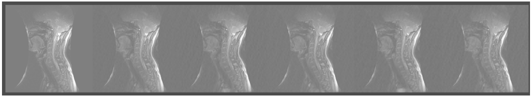
Original sequence



CS−reconstructed sequence



Modified CS reconstructed sequence



99%-energy support $|N_t| \approx 0.07m$, $|\Delta| \approx 0.001m$, $m = 65536$,
used $n = 0.16m$ (for $t > 0$), $n = 0.5m$ (for $t = 0$)

Summary

- Introduced modified-CS for sparse reconstruction problems with partially known support (known part may have error)
- Modified-CS solves

$$\min_{\beta} ||(\beta)_{T^c}||_1 \text{ s.t. } y = A\beta$$

where T is the known part of the support

- Exact reconstruction if $|\Delta| \leq |T|$ and $\delta_{|T|+2|\Delta|} < 1/5$
- *Key app: recursive reconstruction of sparse signal sequences, e.g. real-time dynamic MRI, single-pixel video imaging,...*

Related work

- Similar problem to ours, do not study exact reconstruction
  - Least squares and Kalman filtered CS [Vaswani, ICIP'08, ICASSP'09]
  - Recursive lasso [Angelosante,Giannakis'09]
- Different problem than ours
  - Warm start and homotopy methods [Rozell et al'07, Asif,Romberg'09]
    - use previous recon. and/or homotopy methods to speed up the current optimization
    - *do not use the past to help reduce the number of measurements required for reconstructing the current signal*
  - Reconstruct a single signal from sequentially arriving measurements [Malioutov et al'08, Ghaoui NIPS'08, Asif,Romberg'09]
- Batch or Offline CS: [Wakin et al (video), Gamper et al (MRI), Jung et al (MRI)]

Ongoing and Future Work

- Modified-CS for noisy measurements (sparse KF) or compressible sequences and connections with KF

$$\min_{\beta} \ \gamma||(\beta)_{T^c}||_1 + \tilde{\gamma}||(\beta)_T - (\hat{x}_{t-1})_T||_2^2 + ||y_t - A\beta||_2^2$$

- Stability for signal sequence recon from noisy measurements
  - for a given max. no. of additions per unit time, how slowly should they occur s.t. the error remains bounded at all times?
- Open-ended issues
  - handling deletions from support: currently only using heuristics
  - tight high probability bounds on n needed for modified-CS