Coded Caching Schemes with Reduced Subpacketization from Linear Block Codes

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Content Caching Networks

Content Caching Network

Akamai

edgecast

ChinaCache

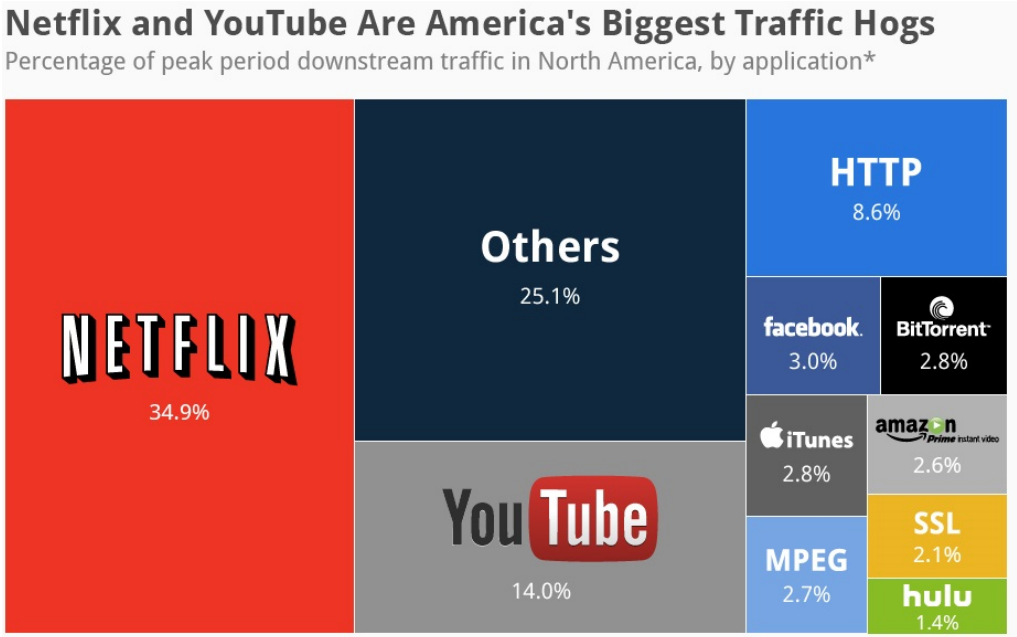
Level(3) COMMUNICATIONS

amazon web services™

fastly

Incapsula

Internet Video Traffic (source: Statista)



- Caching is the technique of choice for reducing video latency and improving end user experience.

One line summary of the talk

Linear block codes (under mild conditions) $\Longrightarrow$ **Caching schemes**

Content Delivery with Caching

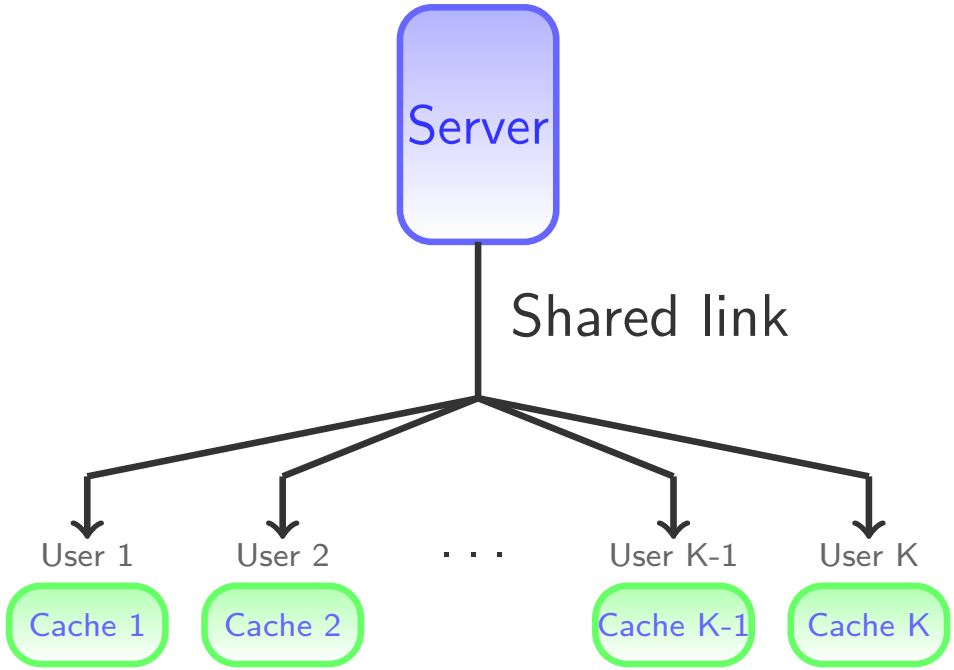
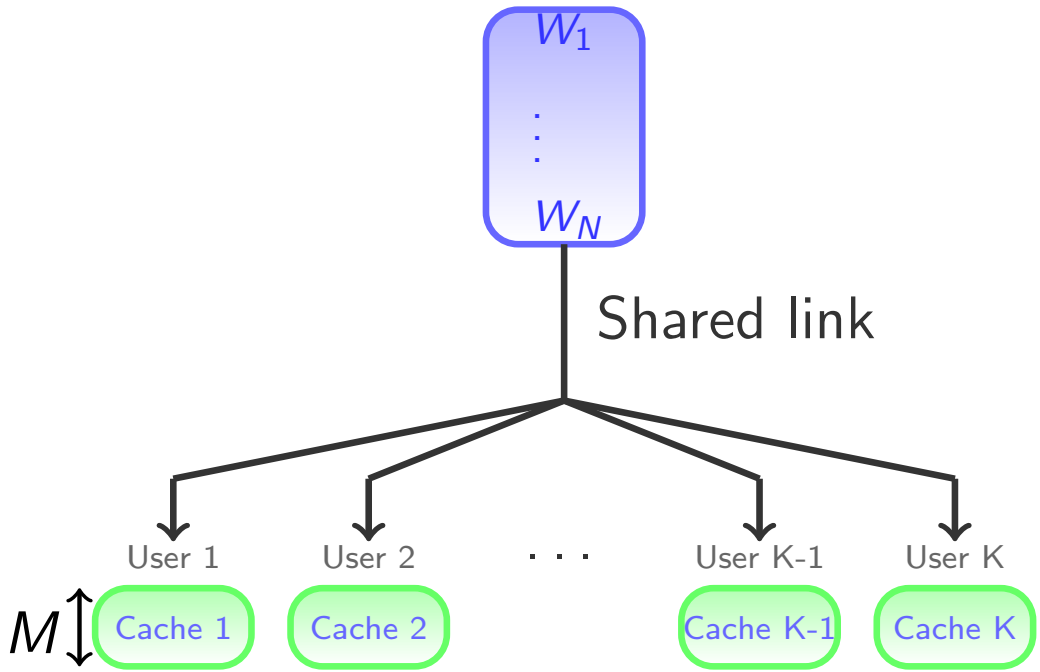


Figure: Content Delivery Network

- Conventional approach: clients cache portions of popular content.
- Coding in the cache and coded transmission from the server are typically not considered.

Coded Caching Formulation *[Maddah-Ali & Niesen '13]*

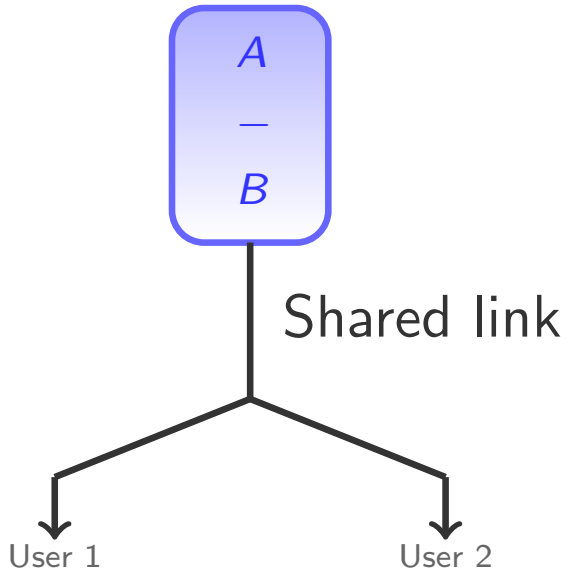
- Server contains a library of N files each of size F subfiles (a subfile is a unit of information, e.g., 1 Mb).
- K users each with a cache of size MF subfiles.
- The i-th user requests file W_{d_i} where $d_i \in \{1, \ldots, N\}$.



Caching Example with $N = K = 2$

Without caching

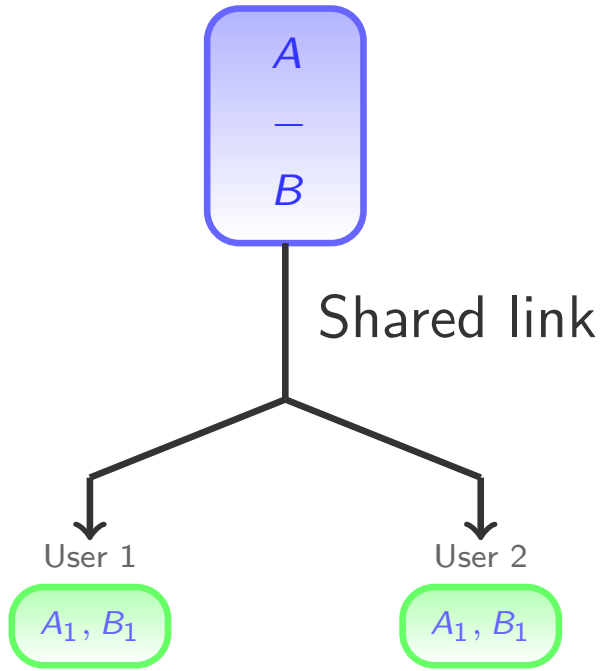
- User 1 requests A and User 2 requests B,
- Server has to send both A and B (two files),



Caching Example with $N = K = 2$

Conventional caching

- Split each file into two equal parts $A = \{A_1, A_2\}, B = \{B_1, B_2\}$.
- Off-peak hours: Users save $M/N = 1/2$ fraction of each file.
- Peak hours: Suppose users 1 and 2 request A and B.
- Server sends file parts A_2 and B_2; equivalent to one file.

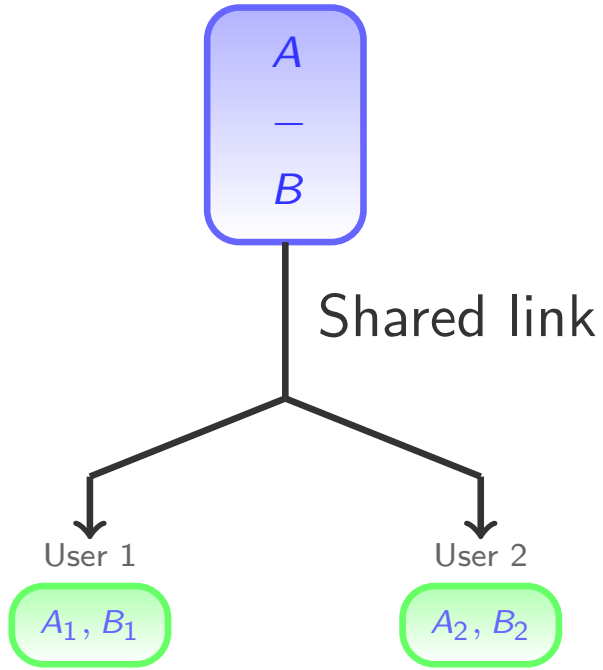


Caching Example with $N = K = 2$

Coded caching

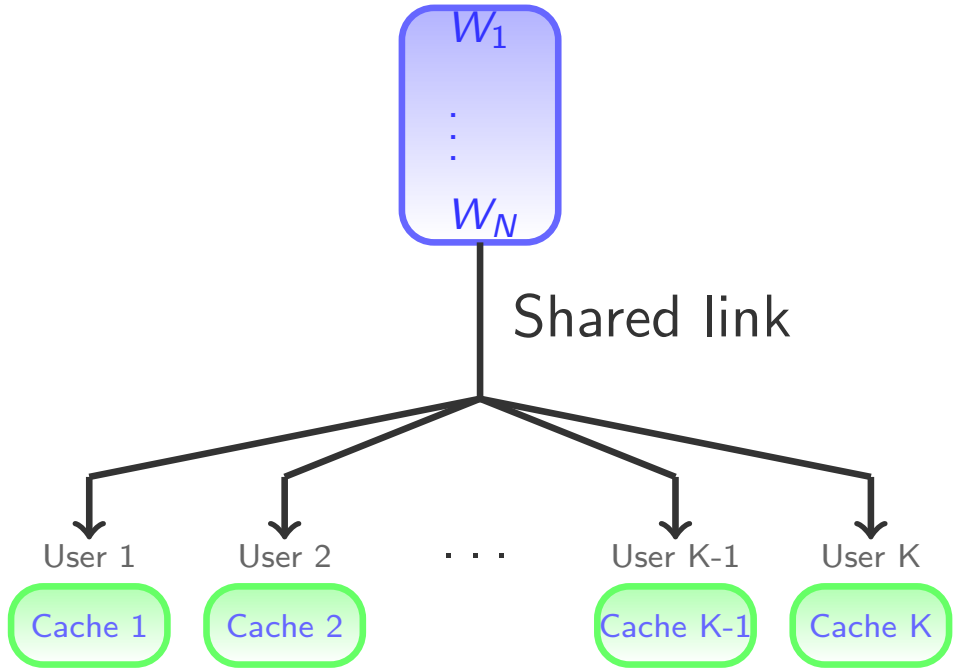
- Split each file into two equal parts $A = \{A_1, A_2\}, B = \{B_1, B_2\}$.
- Off-peak hours: Users cache carefully chosen subsets of the file
  - User 1 caches $Z_1 = \{A_1, B_1\}$.
  - User 2 caches $Z_2 = \{A_2, B_2\}$.
- Peak hours

User 1 Req.	User 2 Req.	Shared link
A	B	$A_2 + B_1$
B	A	$B_2 + A_1$
A	A	$A_2 + A_1$
B	B	$B_2 + B_1$



Over all demand patterns, transmitted rate is at most half a file!!

1. *Placement phase*: Caches are populated. Does NOT depend on actual user requests (off-peak hours).
2. *Delivery phase*: Server transmits a signal of rate RF subfiles over the shared link so that each user's request is satisfied (peak hours).



Potential gains can be huge!

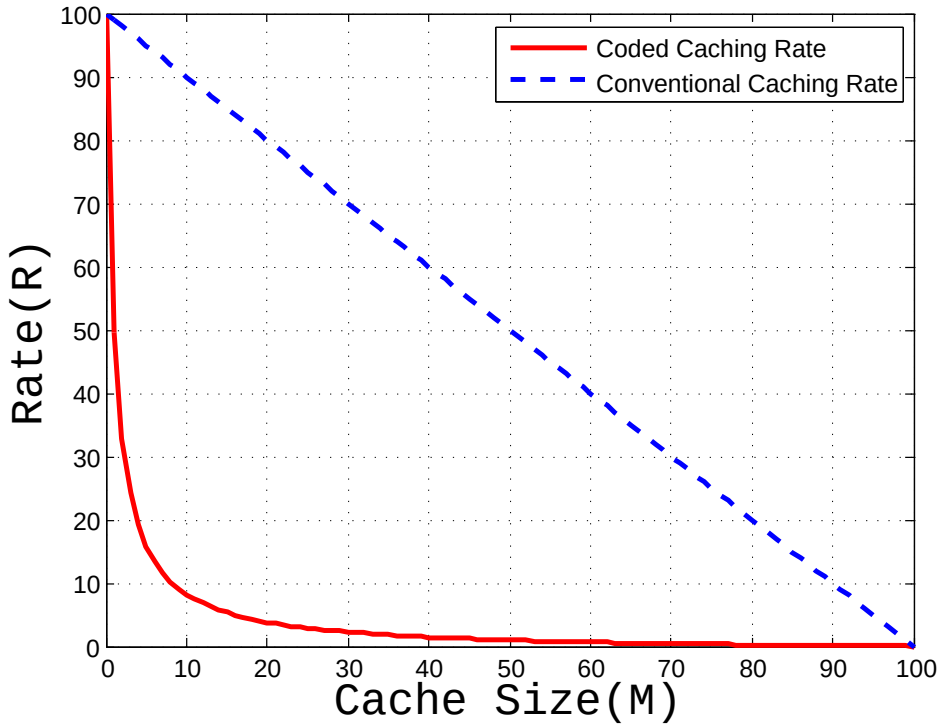
- Number of files N, number of users K, normalized cache per user M. When $N \geq K$

$$R^{\text{uncoded}} = K\left(1 - \frac{M}{N}\right)$$

vs.

$$R^{\text{coded}} = K\left(1 - \frac{M}{N}\right)\frac{1}{1 + \frac{KM}{N}}$$

System with $N = 100, K = 100$



As with most things ...

Source: www.appitierre.com

Practical Issues: Subpacketization Level

- Huge gains of coded caching require each file to be divided into a large number of parts (exponential in problem parameters). Subpacketization level F_s

$$F_s = \binom{K}{KM/N} \approx 2^{KH_2(M/N)}$$

- Example: System with $K = 50$ users, with $M/N = 0.4$, original scheme needs $F_s \approx 10^{14}$.
  - At the bare minimum each file should be of size 100 terabits.
  - In practice, much higher since the basic unit of storage on a computer is a sector.

Related Work

- Problem was first noticed and studied in the context of decentralized coded caching in the work of [Shanmugam et al.].
- For the centralized case, a placement delivery array (PDA) based scheme was proposed in [Yan et al.].
  - Their result can be obtained as a specific case of our result.
- Prior work by us [Tang-Ramamoorthy] also matches the parameters in [Yan et al.].

Related Work

- A hypergraph perspective introduced by [Shangguan et al.]
  - Demonstrates that (asymptotically) constant rate with subpacketization F_s linear in K is impossible.
  - Shows the existence of schemes where subpacketization scales as $\exp(c\sqrt{K})$.
- Work by [Shanmugam et al.] demonstrates connections with induced matchings in Ruzsa-Szemeredi graphs.
  - Shows the existence of schemes where subpacketization is linear but rate scales as K^{δ}, where $0 < \delta < 1$.
- Both these results require rather large values of K and are not applicable in a real-life scenarios.

Designs

Definition

A design is a pair $(X, \mathcal{A})$ such that

1. X is a set of elements called points, and
2. $\mathcal{A}$ is a collection (i.e., multiset) of nonempty subsets of X called blocks, where each block contains the same number of points.

$$X = \{1, 2, 3, 4\} \text{ (Point set)}$$

$$\mathcal{A} = \{\{1, 2\}, \{3, 4\}, \{1, 3\}, \{2, 4\}, \{1, 4\}, \{2, 3\}\} \text{ (Block set)}$$

Resolvable design

Definition

A parallel class $\mathcal{P}$ in a design $(X, \mathcal{A})$ is a subset of disjoint blocks from $\mathcal{A}$ whose union is X. A partition of $\mathcal{A}$ into several parallel classes is called a resolution, and $(X, \mathcal{A})$ is said to be a resolvable design if $\mathcal{A}$ has at least one resolution.

$$X = \{1, 2, 3, 4\} \text{ (Point set)}$$

$$\mathcal{A} = \{\{1, 2\}, \{3, 4\}, \{1, 3\}, \{2, 4\}, \{1, 4\}, \{2, 3\}\} \text{ (Block set)}$$

Each parallel class is a partition of X.

P1	P2	P3
{1, 2}	{1, 3}	{1, 4}
{3, 4}	{2, 4}	{2, 3}

Linear Codes and Resolvable Designs

Let $q = 2$, $k = 3$. Consider a $(3, 2)$ Single Parity Check code over $\mathbb{Z}_2$,

$$\mathbf{G}_{SPC} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{ (Generator Matrix)}$$

Columns of matrix $\mathbf{T}$ list all $2^2 = 4$ codewords.

$$\mathbf{T} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

obtained by computing $\mathbf{c} = \mathbf{u} \cdot \mathbf{G}_{SPC}$ for all possible message vectors $\mathbf{u}$.

Codes and Resolvable Designs

- Points correspond to columns $\{1, \ldots, 4\}$.

$$\mathbf{T} = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

- Blocks are constructed by identifying column index of points in a row.

$$\begin{aligned} [0\ 0\ 1\ 1] &\Rightarrow \mathcal{P}_1 : \{1,2\},\{3,4\} \\ [0\ 1\ 0\ 1] &\Rightarrow \mathcal{P}_2 : \{1,3\},\{2,4\} \\ [0\ 1\ 1\ 0] &\Rightarrow \mathcal{P}_3 : \{1,4\},\{2,3\} \end{aligned}$$

Usage in a coded caching scenario

- Users correspond to blocks (total of six). System with $K = 6$ users and $N \geq K$.

$$\{1,2\} \Rightarrow U_{\{1,2\}} \quad \{3,4\} \Rightarrow U_{\{3,4\}}$$
$$\{1,3\} \Rightarrow U_{\{1,3\}} \quad \{2,4\} \Rightarrow U_{\{2,4\}}$$
$$\{1,4\} \Rightarrow U_{\{1,4\}} \quad \{2,3\} \Rightarrow U_{\{2,3\}}$$

- Subfiles correspond to points (total of four). Split each file into 4 subfiles $W_n = \{W_{n,1}, \ldots, W_{n,4}\}$.
- For original scheme, user corresponds to points (K points), subfile corresponds to $\binom{K}{\frac{KM}{N}}$ blocks (all subsets of size $\frac{KM}{N}$ of $\{1, \cdots, K\}$)

Usage in a coded caching scenario

- Placement Phase: User $U_{\{i,j\}}$ caches for each file W_n in the library,

$$Z_{i,j} = \{W_{n,i}, W_{n,j}\}$$

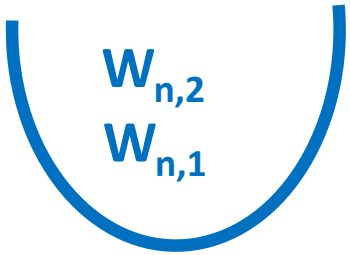
  - For example, user $U_{\{1,2\}}$ caches $\{W_{n,1}, W_{n,2}\}_{n=1}^{N}$.
  - Each user thus caches exactly half of each file. Therefore,

$$\frac{M}{N} = \frac{1}{2}.$$

Usage in a coded caching scenario

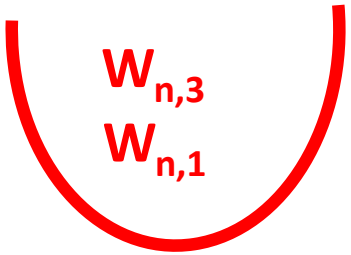
- Delivery Phase: Suppose that each user requests a distinct file (worst case). Thus user $U_{\{i,j\}}$ requests $W_{d_{ij}}$.
- How do we leverage coding in this case?
- Transmit equations that are simultaneously useful to multiple users!!

Example of delivery phase scheme



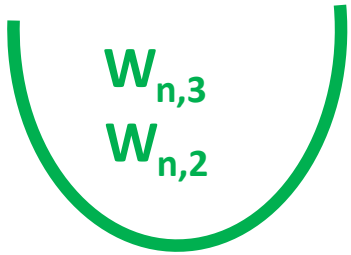
User U_{12}

Wants
$W_{d12,3}$
$W_{d12,4}$



User U_{13}

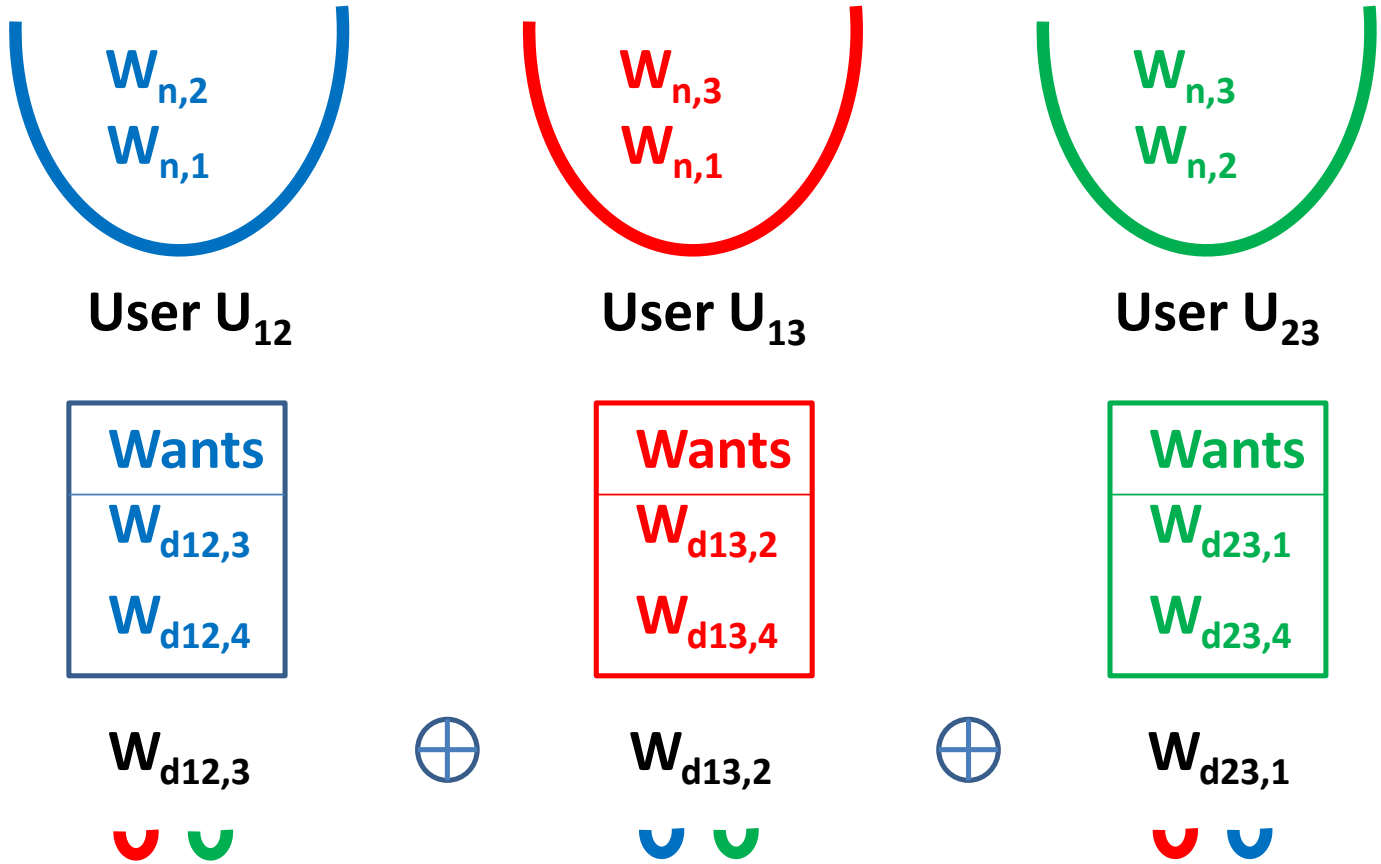
Wants
$W_{d13,2}$
$W_{d13,4}$



User U_{23}

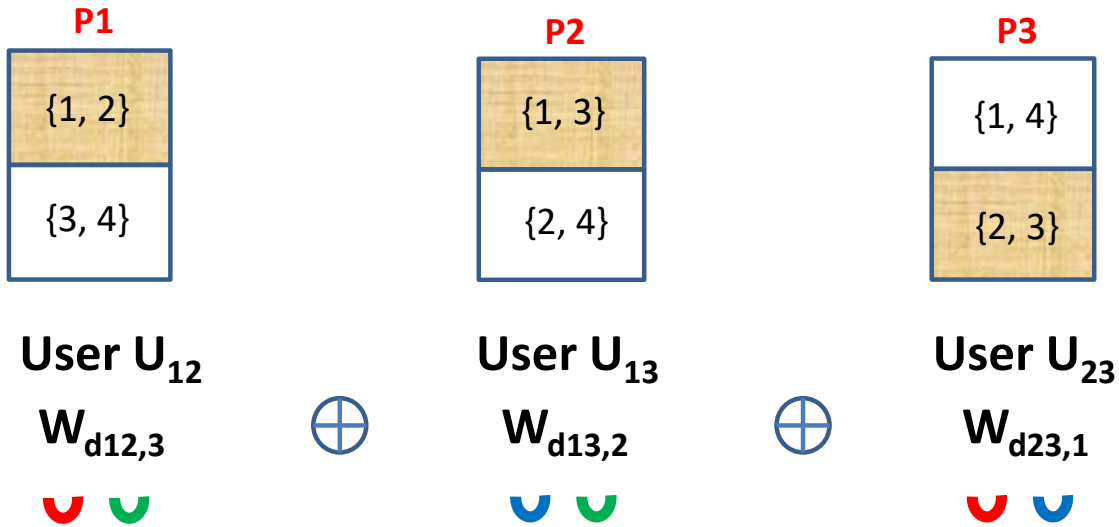
Wants
$W_{d23,1}$
$W_{d23,4}$

Example of delivery phase scheme



Example of delivery phase scheme

Any two blocks from distinct parallel classes have one intersection.



- Any two of blocks share one point that does not contains in the third block. ($\{1,2\},\{1,3\}$ contain point 1 which is not in $\{2,3\}$).
  - The missing subfile of one user in the equation has been cached in another two users.
- Four sets of blocks. Four equations.

Rate Analysis

- Four such equations can be found. Each equation corresponds to $\frac{1}{4}$-th of a file. Total rate is therefore 1 (normalized by file size).
- System has $K = 6$ users, $N \geq K$, $M/N = 1/2$ and a subpacketization level of $F_s = 4$.
- Original scheme would have subpacketization level of $\binom{6}{3} = 20$ and a rate of 0.75.

Rate Analysis

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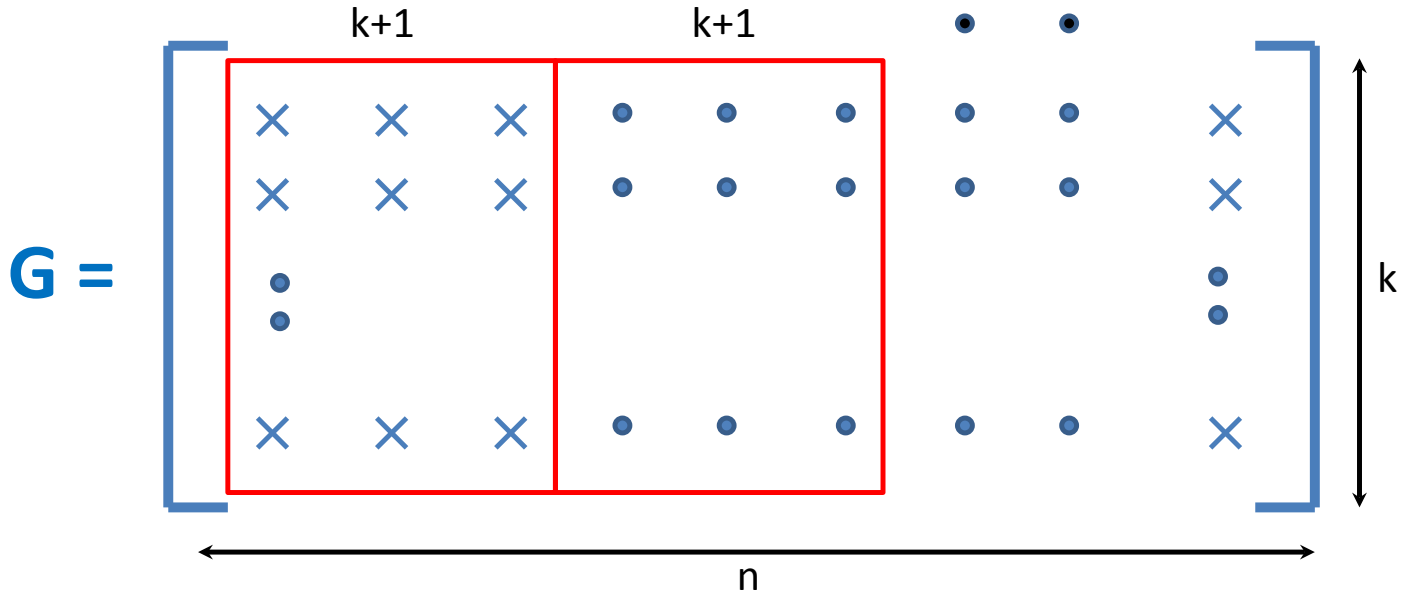
Generalization of approach to a larger class of linear block codes exists....

$(k, k+1)$- Consecutive Column Property (CCP)

- Generator matrix **G** of size $k \times n$ over $GF(q)$. Let z be the *least* integer such that $k+1 \mid nz$
- Recovery Set: Consider the columns of **G** specified by $\mathcal{S}_a$

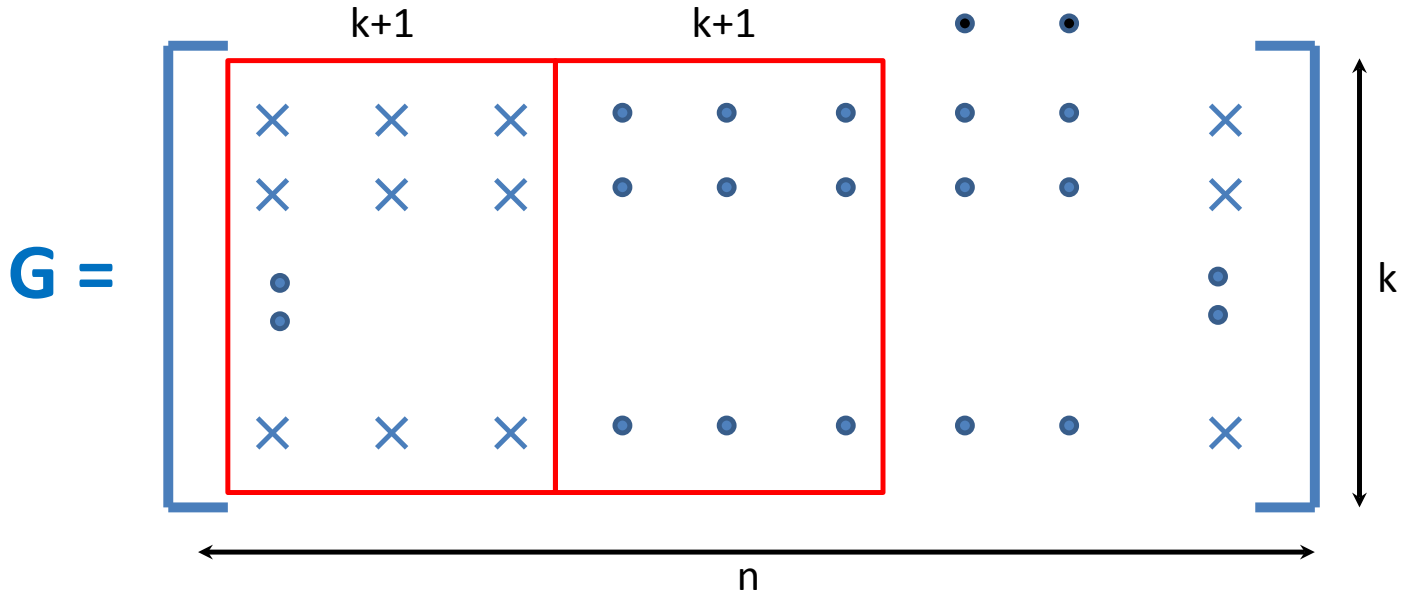
$$\mathcal{S}_a = \{a(k+1), \cdots, a(k+1)+k\} \mod n.$$



$(k, k+1)$- Consecutive Column Property (CCP)

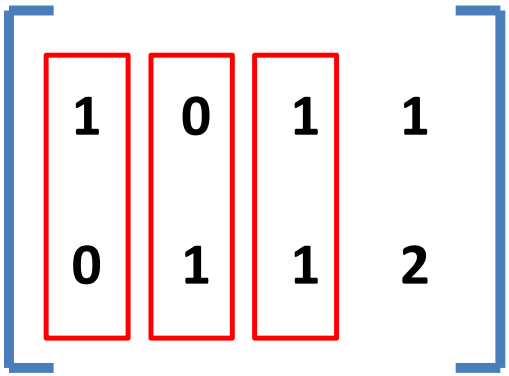
Definition

Consider the submatrices of $\mathbf{G}$ specified by $\mathbf{G}_{\mathcal{S}_a}$ for $0 \leq a \leq \frac{zn}{k+1} - 1$. We say that $\mathbf{G}$ satisfies the $(k, k+1)$-consecutive column property if all $k \times k$ submatrices of each $\mathbf{G}_{\mathcal{S}_a}$ are full rank.



CCP Example - $(4, 2)$ code over $GF(3)$

$$\mathcal{S}_0 = \{0, 1, 2\}$$

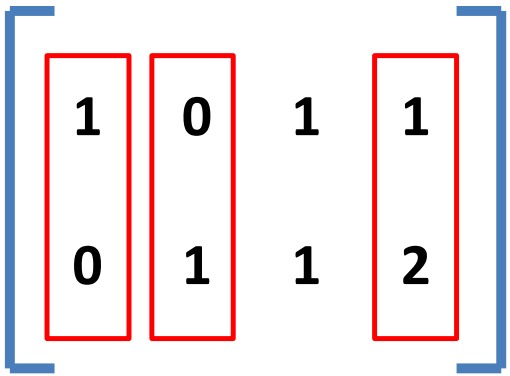


$$\mathbf{G}_{\mathcal{S}_0} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

All 2×2 submatrices are full-rank.

CCP Example - $(4, 2)$ code over $GF(3)$

$$\mathcal{S}_1 = \{3, 1, 0\}$$

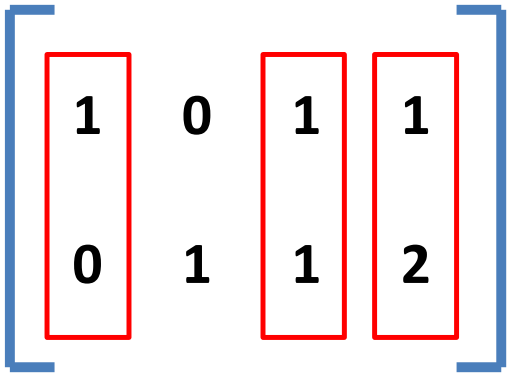


$$\mathbf{G}_{\mathcal{S}_1} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

All 2×2 submatrices are full-rank.

CCP Example - $(4, 2)$ code over $GF(3)$

$$\mathcal{S}_2 = \{2, 3, 0\}$$

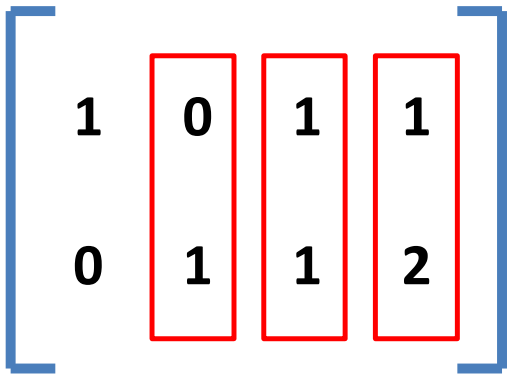


$$\mathbf{G}_{\mathcal{S}_2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

All 2×2 submatrices are full-rank.

CCP Example - $(4, 2)$ code over $GF(3)$

$$\mathcal{S}_3 = \{1, 2, 3\}$$



$$\mathbf{G}_{\mathcal{S}_3} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

All 2×2 submatrices are full-rank.

Usage in coded caching scenario

$$\mathbf{T} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 \\ 0 & 2 & 1 & 1 & 0 & 2 & 2 & 1 & 0 \end{bmatrix}.$$

- Point set

$$X = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}.$$

- Block set (Parallel class)

$$P_0 : \{\{0, 1, 2\}, \{3, 4, 5\}, \{6, 7, 8\}\}$$
$$P_1 : \{\{0, 3, 6\}, \{1, 4, 7\}, \{2, 5, 8\}\}$$
$$P_2 : \{\{0, 5, 7\}, \{1, 3, 8\}, \{2, 4, 6\}\}$$
$$P_3 : \{\{0, 4, 8\}, \{2, 3, 7\}, \{1, 5, 6\}\}$$

Usage in coded caching scenario

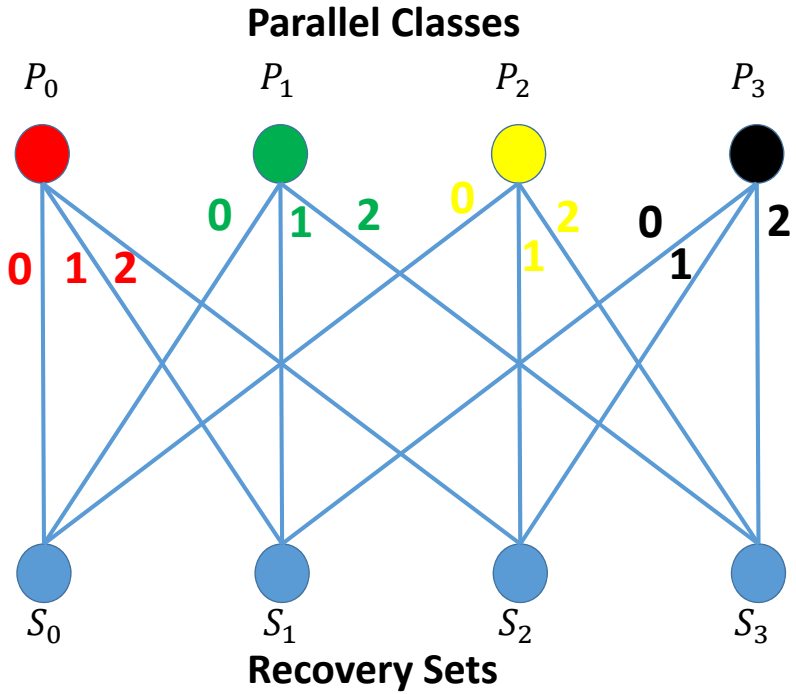
$$X = \{0,1,2,3,4,5,6,7,8\}$$
$$P_0 : \{\{0,1,2\},\{3,4,5\},\{6,7,8\}\}$$
$$P_1 : \{\{0,3,6\},\{1,4,7\},\{2,5,8\}\}$$
$$P_2 : \{\{0,5,7\},\{1,3,8\},\{2,4,6\}\}$$
$$P_3 : \{\{0,4,8\},\{2,3,7\},\{1,5,6\}\}$$

- Users correspond to blocks.
- Subfiles correspond to points and another index $\in \{0,\ldots,z-1\}$.

Example

User U_{012} caches $W^s_{n,0}, W^s_{n,1}, W^s_{n,2}$ for $s = 0,1,2$.

Delivery Phase Algorithm



- Each parallel class $\mathcal{P}_i$ recovers missing subfiles with a specific superscript for each user.
  - Similar to single-parity-check (SPC) code.
- Recovery set bipartite graph formalizes the idea.
- Recovery set $\mathcal{S}_0$: Constructive procedure that picks a user from each parallel class in $\mathcal{P}_0, \mathcal{P}_1$ and $\mathcal{P}_2$.

Delivery Phase Algorithm

- With some work, we can show that the number of user is $K = nq$, cache size is $\frac{M}{N} = \frac{1}{q}$, the rate of our scheme is

$$R^* = \frac{(q-1)n}{k+1}.$$

  Then coded gain is

$$g^* = k + 1.$$

- The subpacketization level is

$$F_s^* = q^k z$$

- Important fact: Both coded gain g^* and subpacketization level F_s^* increase with larger k!!

Delivery Phase Algorithm

- Scheme for $\frac{M}{N} = \frac{1}{q}$ can be converted into a scheme for $\frac{M}{N} = 1 - \frac{k+1}{nq}$ with

$$R^* = \frac{k+1}{(q-1)n}$$

$$F_s^* = (q-1)q^k \frac{zn}{k+1}$$

Some Linear Codes satisfying the CCP

- MDS code over $GF(q)$ of dimension $k \times n$
- Satisfies a stronger condition. *Any $k \times k$ submatrix is full-rank.*
  - Drawback: MDS codes typically need $q + 1 \geq n$.
  - Thus, with increasing n, the cached fraction $M/N = 1/q$ keeps decreasing.

Some Linear Codes satisfying the CCP

- A large class of cyclic codes!!

$$\mathbf{G} = \begin{bmatrix} g_0 & g_1 & \cdot & \cdot & \cdot & g_{n-k} & 0 & \cdot & \cdot & 0 \\ 0 & g_0 & g_1 & \cdot & \cdot & \cdot & g_{n-k} & 0 & \cdot & 0 \\ \vdots & & & & & & & & & \vdots \\ 0 & 0 & \cdot & 0 & g_0 & g_1 & \cdot & \cdot & \cdot & g_{n-k} \end{bmatrix}$$

- Any $k \times n$ generator matrix of a cyclic code over $GF(q)$ (if it exists) is such that
  - Any k consecutive columns are linearly independent.
  - We need slightly more... All $k \times k$ submatrices of $k+1$ consecutive columns are full-rank.
  - Equivalent to conditions on the generator polynomial of the cyclic code. Several examples of such codes can be found.

Some Linear Codes satisfying the CCP

- Constructions leveraging properties of smaller base matrices
  Eg. $\mathbf{A}$ satisfies the CCP, $\mathbf{G} = \mathbf{A} \otimes \mathbf{I}_{a \times a}$ satisfies the CCP.
- Some linear codes over $\mathbb{Z}$ mod q satisfy the CCP (q is not necessarily prime or prime power)
  - E.g. $(k+1, k)$ single parity check (SPC) code over $\mathbb{Z}$ mod q.
  - The conditions under which a resolvable design can be obtained by forming the matrix $\mathbf{T}$ are a little more involved.
- Some linear codes satisfy (k, α) CCP, $\alpha \leq k$.
  - α columns in each recovery set of size α have full rank.
  - It yields a lower coded gain $g = \alpha \leq k+1$ but larger range of parameters.
  - E.g. All cyclic codes satisfy (k, k)-CCP.

Example: $K = nq = 60$ users, $M/N = 1/q = 1/5$

k	F_s	g	R	Notes
-	1.4×10^{12}	13	3.69	Maddah-Ali & Niesen
11	4.9×10^{7}	12	4	
10	-	-	-	$(k, k+1)$-CCP code not exist
9	6.8×10^{6}	10	4.8	
8	1.2×10^{6}	9	5.3	
7	1.5×10^{5}	8	6	
6	10^{5}	7	6.85	
5	3125	6	8	
4	3125	5	9.6	
3	125	4	12	
2	25	3	16	
1	5	2	24	

Table: List of k values and corresponding F_s, g and R

Parameters: $K = nq$ users, $M/N = 1/q$ cache fraction, $F_s = q^k z$ subpacketization level

- Rate comparison (loss is tunable with choice of (n, k) code)

$$\frac{R^*}{R^{MN}} = \frac{n+1}{k+1}$$

- Subpacketization level (exponential improvement)

$$\frac{F_s^*}{F_s^{MN}} \approx q^{k-n} z \left(\frac{q-1}{q} \right)^{n(q-1)}.$$

Parameters: $K = nq$ users, $M/N = 1/q$ cache fraction, $F_s = q^k z$ subpacketization level

- Comparisons when rates of both schemes are almost the same is somewhat harder ...
- Use memory-sharing in Maddah-Ali & Niesen.

Parameters: $K = nq$ users, $M/N = 1/q$ cache fraction, $F_s = q^k z$ subpacketization level

- $(9, 5)$ linear block code over $GF(2)$ that satisfies CCP.

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

  Parameters: $K = 9 \times 2 = 18$, $\frac{M}{N} = \frac{1}{2}$, $R^* = \frac{3}{2}$ and $F_s^* = 64$.

- Maddah-Ali & Niesen approach would need memory-sharing between $\frac{M_1}{N_1} = \frac{2}{9}$ and $\frac{M_2}{N_2} = \frac{7}{9}$.
  $R^{MN} \approx 3/2$. However, the subpacketization level is
  $F_s^{MN} = \binom{K}{KM_1/N_1} + \binom{K}{KM_2/N_2} = 6120$. (6120 *vs* 64)

Summary

- Subpacketization is an important issue in coded caching.
- Our work establishes a simple and useful connection between classes of linear codes and placement schemes in coded caching.
- The schemes are easy to construct and analyze.
- Allow us to obtain schemes with exponentially smaller subpacketization levels compared to the original scheme.

Tang & R., "*Coded Caching Schemes with Reduced Subpacketization from Linear Block Codes*",
IEEE Trans. on Information Theory, May 2018.

Several Open Questions

- The subpacketization level of our construction still scales exponentially with the problem parameters.
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- Constructions that are flexible with respect to the cache fractions that they support.