# Constructions of fractional repetition codes from combinatorial designs 

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#### Abstract

We consider the design of regenerating codes for distributed storage systems at the minimum bandwidth regeneration (MBR) point. Our codes consist of an outer MDS code followed by an inner fractional repetition (FR) code (introduced in prior work). In these systems a failed node can be repaired by simply downloading packets from surviving nodes. We present constructions that use the Kronecker product to construct new fractional repetition codes from existing codes. We demonstrate that an infinite family of codes can be generated by considering the Kronecker product of two Steiner systems that have the same storage capacity. The resultant code inherits its normalized repair bandwidth from the storage capacity of the original Steiner systems and has the maximum level of failure resilience possible. We also present some properties of the Kronecker product of resolvable designs and the corresponding file sizes.


## I. Introduction

Distributed storage systems (DSS) are used to store large amounts of data in a distributed manner. Applications of DSS can be found in social networking websites, video streaming websites and cloud storage systems. In these systems the individual storage nodes may be unreliable. Ensuring the reliability of the system requires the introduction of redundancy through replication of the data and/or erasure coding. The current generation of DSS also have the requirement that upon failure of a node, the failed node can be appropriately regenerated so that the system continues to work as before. It is evident that there are several issues that need to be considered with respect to the regenerating process. For instance, one would like the regenerating process to be fast. For this purpose we would like to minimize the data that needs to be downloaded from the surviving nodes. The problem of regenerating codes was introduced in the work of Dimakis et al. [1] and has been the subject of much investigation in recent years (see [2] and its references).

A distributed storage system consists of $n$ storage nodes, each of which stores $\alpha$ packets (or equivalently symbols). A given user, also referred to as the data collector needs to have the ability to reconstruct the stored file by contacting any $k$ nodes; this is referred to as the maximum distance separability (MDS) property of the system. When a node fails, the new node needs to connect to any $d \geq k$ surviving nodes and download $\beta$ packets from each of them for a total

[^0]repair bandwidth of $\gamma=d \beta$ packets. Thus, the system has a repair degree of $d$, normalized repair bandwidth $\beta$ and total repair bandwidth $\gamma$. The new DSS should continue to have the MDS property. Two types of repair processes can be considered - functional and exact repair. The original work of [1] considered functional repair where the new node needs to be functionally equivalent to the failed node. An informationtheoretic tradeoff between the amount of data stored at each node (storage capacity) and the amount of data that needs to be downloaded for regenerating a failed node (repair bandwidth) was determined. In the case of exact repair [3] the new node needs to be an exact copy of the failed node. For this case, constructions are typically known for either the (i) minimum bandwidth regenerating (MBR) point where the repair bandwidth, $\gamma$ is minimum, or for the (ii) minimum storage regenerating (MSR) point where the storage per node, $\alpha$ is minimum.

Several works have considered alternate metrics for the regeneration process. In addition to the repair bandwidth, one can also consider code constructions (called local codes [4][5][6]) that minimize the number of nodes that are contacted in the regeneration process. In these codes the repair degree $d$ is strictly smaller than $k$. It has also been recognized that the regeneration process of some MBR/MSR code constructions can be memory intensive and introduce undesirable latencies in the repair process [7]. Accordingly, another line of work has considered regenerating codes that have the exact and uncoded repair property [8]. Specifically, in these codes a failed node is exactly recovered by simply downloading symbols from the surviving nodes, i.e., the regeneration does not involve any decoding. It is to be noted that for these codes the DSS only has the property that the failed node can be recovered by contacting a specific set (or sets) of $d$ nodes.

In prior work, we considered several classes of codes with exact and uncoded repair that have the outer MDS code followed by an inner fractional repetition code architecture proposed in [8]. Specifically in [9], we demonstrated codes with $d \geq k$ from resolvable designs [10] that can be constructed based on affine resolvable designs, Hadamard designs, lattice codes and mutually orthogonal Latin squares. Subsequently in [11], we presented techniques for the design of systems that have the locality property, i.e., $d<k$.

In this work, we propose new constructions that are based on computing the Kronecker product of existing codes. We demonstrate that an infinite family of codes can be generated by considering the Kronecker product of two Steiner systems that have the same storage capacity. The resultant code inherits its normalized repair bandwidth from the storage capacity of the original Steiner systems and has the maximum level of failure resilience possible. We also present some properties of the Kronecker product of resolvable designs and the corresponding file sizes.

## II. Background and Problem Formulation

We begin by providing an example (that first appeared in [3]) that illustrates the basic underlying concepts in a fractional repetition code with $(n, k, d)=(5,3,4)$.


Fig. 1. A DSS with $(n, k, d, \alpha)=(5,3,4,4)$. Each node contains a subset of size 4 of the packets from $\left\{b_{1}, \ldots, b_{10}\right\}$. Node $V_{1}$ for instance contains symbols $b_{i}, i=1, \ldots, 4$.

Example 1: Consider a file of $\mathcal{M}=9$ packets $\left(a_{1}, \ldots, a_{9}\right) \in \mathbb{F}_{q}^{9}$ that needs to stored on the DSS. We use a $(10,9)$ MDS code that outputs 10 packets $b_{i}=a_{i}, i=1, \ldots, 9$ and $b_{10}=\sum_{i=1}^{9} a_{i}$. The coded packets $b_{1}, \ldots, b_{10}$ are placed on $n=5$ storage nodes as shown in Fig. 1. This placement specifies the inner fractional repetition code. It can be observed that each $b_{i}$ is repeated $\rho=2$ times and the total number of symbols $\theta=10$. Any user who contacts any $k=3$ nodes can recover the file (using the MDS property). Moreover, it can be verified that if a node fails, one packet each can be downloaded from the four surviving nodes, i.e., $\beta=1$ and $d=4$, so that $\gamma=4$.

Thus, the approach uses an MDS code to encode a file consisting of a certain number of symbols. Let $\theta$ denote the number of encoded symbols. Copies of these symbols are placed on the $n$ nodes such that each symbol is repeated $\rho$ times and each node contains $\alpha$ symbols. Moreover, if a given node fails, it can be exactly recovered by downloading $\beta$ packets from some set of $d$ surviving nodes, for a total repair bandwidth of $\gamma=d \beta$. It is to be noted that in this case $\alpha=\gamma$, i.e., these schemes operate at the MBR point. In the example above, $\beta=1$, so that $\alpha=d$. One can also
consider systems with $\beta>1$ in general. A simple way to do this is by replicating the symbols in the storage system. The resultant DSS has the parameters $(n, k, d, \beta \alpha)$ with $\beta>1$. However there are infinite families of FR codes with $\beta>1$ which cannot be obtained this way. For instance, it can be shown that some of the constructions of FR codes with $\beta>1$ that appeared in our prior work [9] cannot be obtained by a simple replication. Let $[n]$ denote the set $\{1,2, \ldots, n\}$.

Definition 1: Let $\Omega=[\theta]$ and $V_{i}, i=1, \ldots, d$ be subsets of $\Omega$. Let $V=\left\{V_{1}, \ldots, V_{d}\right\}$ and consider $A \subset \Omega$ with $|A|=$ $d \beta$. We say that $A$ is $\beta$-recoverable from $V$ if there exists $B_{i} \subseteq V_{i}$ for each $i=1, \ldots, d$ such that $B_{i} \subset A,\left|B_{i}\right|=\beta$ and $\cup_{i=1}^{d} B_{i}=A$.

A fractional repetition (FR) code $\mathcal{C}=(\Omega, V)$ for a ( $n, k, d, \alpha)$-DSS with repetition degree $\rho$ and normalized repair bandwidth $\beta=\alpha / d$ ( $\alpha$ and $\beta$ are positive integers) is a set of $n$ subsets $V=\left\{V_{1}, \ldots, V_{n}\right\}$ of a symbol set $\Omega=[\theta]$ with the following properties.
(a) The cardinality of each $V_{i}$ is $\alpha$.
(b) Each element of $\Omega$ belongs to $\rho$ sets in $V$.
(c) Let $V^{\text {surv }}$ denote any $\left(n-\rho_{\text {res }}\right)$ sized subset of $V$ and $V^{\text {fail }}=V \backslash V^{\text {surv }}$. Each $V_{j} \in V^{\text {fail }}$ is $\beta$-recoverable from some $d$-sized subset of $V^{\text {surv }}$.
The value of $\rho_{\text {res }}$ is a measure of the resilience of the system to node failures, under the constraint of exact and uncoded repair. The file size is given by

$$
\mathcal{M}=\min _{I \subset[n],|I|=k}\left|\cup_{i \in I} V_{i}\right|,
$$

where $[n]=\{1, \ldots, n\}$ and the code rate is defined as $R_{\mathcal{C}}=$ $\frac{\mathcal{M}}{n \alpha}$.
The incidence matrix of a FR code $\mathcal{C}=(\Omega, V)$ where $\Omega=$ [ $\theta$ ] and $V=\left\{V_{1}, V_{2}, \ldots, V_{n}\right\}$ is the $\theta \times n\{0,1\}$-matrix $N$ defined by

$$
N_{i, j}= \begin{cases}1, & \text { if } i \in V_{j} \\ 0, & \text { otherwise }\end{cases}
$$

For a FR code with incidence matrix $N$, the transpose FR code refers to the code specified by $N^{T}$. In the transposed code, the roles of the storage nodes and the symbols are reversed.

The original work of [8] and the subsequent work [12] only considered FR codes where the repair degree $d \geq k$. It can be seen that the constraint $d \geq k$ is essential in the model of [1] as it is required that a failed node can be recovered from any set of $d$ nodes. However, in the current setup we consider table based recovery, where the failed node is recovered by contacting a certain set of $d$ surviving nodes. Accordingly, in our setup it is possible that $d<k$. In prior work [11] (see also [13]) we demonstrated constructions of FR codes, called "local" FR codes where $d<k$. As we shall see, the Kronecker product technique discussed in the present paper can also generate local FR codes.

In prior work [9], we proposed several constructions of fractional repetition codes based on affine resolvable designs, Hadamard designs, lattice codes and mutually orthogonal Latin
squares where the recovery degree $d \geq k$. These constructions fall under the class of resolvable FR codes that are defined below.

Definition 2: Resolvable $F R$ code. Let $\mathcal{C}=(\Omega, V)$ where $V=\left\{V_{1}, \ldots, V_{n}\right\}$ be a FR code. A subset $P \subset V$ is said to be a parallel class if for $V_{i} \in P$ and $V_{j} \in P$ with $i \neq j$ we have $V_{i} \cap V_{j}=\emptyset$ and $\cup_{\left\{j: V_{j} \in P\right\}} V_{j}=\Omega$. A partition of $V$ into $r$ parallel classes is called a resolution. If there exists at least one resolution then the code is called a resolvable fractional repetition code.
In this work we also consider Kronecker product constructions where the constituent codes are resolvable in nature.

## III. Some characteristics of Kronecker product of FR CODES

In the discussion below we demonstrate that incidence matrices with appropriate parameters can be combined via operations such as the Kronecker product to obtain new matrices (equivalently FR codes) with a newer set of parameters.

Definition 3: If $A$ is an $m$-by- $r$ matrix and $B$ is a $p$-by- $q$ matrix, then the Kronecker product $A \otimes B$ is the $m p$-by-r $q$ block matrix

$$
\left(\begin{array}{cccc}
a_{11} B & a_{12} B & \cdots & a_{1 r} B \\
a_{21} B & a_{22} B & \cdots & a_{2 r} B \\
\vdots & \vdots & \vdots & \vdots \\
a_{m 1} B & a_{m 2} B & \cdots & a_{m r} B
\end{array}\right)
$$

Let $N_{1}$ and $N_{2}$ be two incidence matrices of FR $\operatorname{codes} \mathcal{C}_{1}=$ $\left(\Omega_{1}, V_{1}\right)$ and $\mathcal{C}_{2}=\left(\Omega_{2}, V_{2}\right)$ with parameters $\left(n_{1}, \theta_{1}, \alpha_{1}, \rho_{1}\right)$ and $\left(n_{2}, \theta_{2}, \alpha_{2}, \rho_{2}\right)$ respectively. Let $c_{1}, \cdots, c_{n_{1}}$ be the $n_{1}$ columns of $N_{1}$ and $d_{1}, \cdots, d_{n_{2}}$ be the $n_{2}$ columns $N_{2}$. A new FR code with incidence matrix $\bar{N}$ can be obtained as follows.

$$
\bar{N}=\left[\begin{array}{llll}
N_{1} \otimes d_{1} & N_{1} \otimes d_{2} & \cdots & N_{1} \otimes d_{n_{2}}
\end{array}\right]
$$

It can be seen that $\bar{N}$ is equivalent to the Kronecker product of $N_{1}$ and $N_{2}$ upon appropriate permutation of its rows and columns.

As a simple example we can obtain a DSS obtained by replicating the symbols of another DSS via the Kronecker product as follows. Let $N$ be the incidence matrix of a FR code with parameters $(n, \theta, \alpha, \rho)$. Let $\mathbb{1}$ be the $m \times 1$ all-ones column vector. The FR code obtained from $\bar{N}=N \otimes \mathbb{1}$ has parameters ( $n, \theta m, \alpha m, \rho$ ).

More generally, we can obtain FR codes by taking the Kronecker product of existing FR codes. This is best illustrated by the following example that generates a local FR code by taking the Kronecker product of a FR code with itself.

Example 2: Locally recoverable code using Kronecker product technique. Let $\mathcal{C}=(\Omega, V)$ be a FR code with $\Omega=$ $\{1,2,3\}$ and $V=\left\{V_{1}=\{1,2\}, V_{2}=\{2,3\}, V_{3}=\{1,3\}\right\}$. The incidence matrix of the code obtained from $\bar{N}=N \otimes N$

| 12 |
| :---: |
| 23 |
| 13 |$\diamond$| 12 |
| :---: |
| 13 |$\rightarrow$| 1245 | 4578 | 1278 |
| :---: | :---: | :---: |
| 2356 | 5689 | 2389 |
| 1346 | 4679 | 1379 |

Fig. 2. A failed node can be recovered by contacting 2 nodes and downloading 2 packets from each. The code is resilient up to 3 failures and the file size is 8 .
is given below and the storage nodes are shown in Figure 2.

$$
\left[\begin{array}{lllllllll}
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1
\end{array}\right] .
$$

Suppose that the outer MDS code has parameters $(9,8)$, so that $\theta=9, \mathcal{M}=8$. It can be observed in this construction that the file can be recovered by contacting any four nodes, so that $k=$ 4 and that a failed node can be recovered by contacting two nodes and downloading two packets from each of them. Thus the DSS is specified with parameters $(n, k, d, \alpha)=(9,4,2,4)$. As $d<k$, the code is locally recoverable.

An infinite family of FR codes can be obtained by considering Kronecker products of combinatorial structures known as Steiner systems [10].
Definition 4: A $S(t, \alpha, \theta)$ Steiner system is a set $\Omega$ of $\theta$ elements and a collection of subsets of $\Omega$ of size $\alpha$ called blocks such that any $t$ subset of the symbol set $\Omega$ appears in exactly one of the blocks.
The majority of constructions of Steiner systems are known for the case of $t=2$. For these system any two symbols are contained in exactly one block. This also means that any two blocks have at most one symbol in common. It follows that for a Steiner system based FR code, we have $\beta=1$. We show below however, that the Kronecker product of two Steiner systems with the same storage capacity $(\alpha)$ inherits its normalized repair bandwidth (i.e., its $\beta$ ) from the storage capacity of the original systems. Appropriate choice of parameters also allows us to maintain the system resilience as high as possible.
Lemma 1: Let $N_{1}$ and $N_{2}$ be incidence matrices of two Steiner systems with parameters $\left(n_{1}, \theta_{1}, \alpha, \rho_{1}\right)$ and $\left(n_{2}, \theta_{2}, \alpha, \rho_{2}\right)$. Assume that the FR code obtained from $\bar{N}=$ $N_{1} \otimes N_{2}$ has normalized repair bandwidth $\beta=\alpha$. Then the FR code $\bar{N}$ is resilient up to $\rho_{1} \rho_{2}-1$ failures.

Proof: Define $\mathcal{N}\left(c_{i}\right)\left(\mathcal{N}\left(d_{j}\right)\right)$ to be the set of storage nodes in $N_{1}\left(N_{2}\right)$ that have exactly one symbol in common with $c_{i}\left(d_{j}\right)$. As $N_{1}$ and $N_{2}$ are Steiner systems, two nodes have at most one symbol in common.

In the discussion below we show that if there are at most $\rho_{1} \rho_{2}-1$ failures, we can recover all the nodes. We proceed by contradiction, i.e., assume that there exists a set of failed nodes $F^{*}$ in $\bar{N}$ with $\left|F^{*}\right|=\rho_{1} \rho_{2}-1$. Suppose that there is a failed node $c_{i} \otimes d_{j} \in F^{*}$ that cannot be recovered. Note that $\beta=\alpha$. Thus, we need to download $\alpha$ symbols each from the surviving nodes, i.e., we need to consider nodes in $\bar{N}$ that have an overlap of $\alpha$ with $c_{i} \otimes d_{j}$.

Our first observation is that only the nodes in $\mathcal{N}\left(c_{i}\right) \otimes d_{j}$ and $c_{i} \otimes \mathcal{N}\left(d_{j}\right)$ are useful for recovering $c_{i} \otimes d_{j}$. To see this consider a node $c_{i}^{\prime} \otimes d_{j}^{\prime}$ in $\bar{N}$ such that it does not belong to $\mathcal{N}\left(c_{i}\right) \otimes d_{j}$ or $c_{i} \otimes \mathcal{N}\left(d_{j}\right)$. If $c_{i}^{\prime}=c_{i}$, then $d_{j}^{\prime} \notin \mathcal{N}\left(d_{j}\right)$, i.e., $\left(c_{i}^{\prime} \otimes d_{j}^{\prime}\right)^{t}\left(c_{i} \otimes d_{j}\right)=0$; a similar argument holds when $c_{i}^{\prime} \notin \mathcal{N}\left(c_{i}\right), d_{j}^{\prime}=d_{j}$. Otherwise $\left(c_{i}^{\prime} \otimes d_{j}^{\prime}\right)^{t}\left(c_{i} \otimes d_{j}\right)=\left(c_{i}^{\prime t} c_{i}\right) \otimes$ $\left(d_{j}^{\prime t} d_{j}\right)$ can be at most 1 . Thus, only the nodes in $\mathcal{N}\left(c_{i}\right) \otimes d_{j}$ and $c_{i} \otimes \mathcal{N}\left(d_{j}\right)$ are useful for reconstructing $c_{i} \otimes d_{j}$.

Next, note that $c_{i}\left(d_{j}\right)$ can be expressed as the sum of $\alpha$ unit vectors of length $\theta_{1}\left(\theta_{2}\right)$. Let $e_{k}$ denote the unit vector with a one in the $k$-th location. Thus, $c_{i}=\sum_{k \in I_{1}} e_{k}$, where $I_{1} \subset\left[\theta_{1}\right]$ and $d_{j}=\sum_{l \in I_{2}} e_{l}$ where $I_{2} \subset\left[\theta_{2}\right]$. Thus, the overlap between $c_{i} \otimes d_{j}$ and $\mathcal{N}\left(c_{i}\right) \otimes d_{j}$ can be expressed as $e_{k} \otimes d_{j}$ for some $k \in I_{1}$. A similar statement holds for the overlap between $c_{i} \otimes d_{j}$ and $c_{i} \otimes \mathcal{N}\left(d_{j}\right)$. Our next observation is that when we reconstruct $c_{i} \otimes d_{j}$, we can either download symbols from $\mathcal{N}\left(c_{i}\right) \otimes d_{j}$ or from $c_{i} \otimes \mathcal{N}\left(d_{j}\right)$ but not both. Indeed, for $k \in I_{1}, l \in I_{2}$, we have $\left(c_{i} \otimes e_{l}\right)^{t}\left(e_{k} \otimes d_{j}\right)=1$. Thus, if we download symbols from both $\mathcal{N}\left(c_{i}\right) \otimes d_{j}$ and from $c_{i} \otimes \mathcal{N}\left(d_{j}\right)$, then we will need to download strictly more than $\alpha^{2}$ symbols for reconstructing $c_{i} \otimes d_{j}$.

Note that there are $\rho_{1}$ copies of each $e_{k} \otimes d_{j}$, where $k \in I_{1}$. If there is at least one copy of $e_{k} \otimes d_{j}$, for all $k \in I_{1}$ available in the surviving nodes, then it is clear that $c_{i} \otimes d_{j}$ can be recovered by downloading copies of each $e_{k} \otimes d_{j}$ from the surviving nodes. Likewise, there are $\rho_{2}$ copies of each $c_{i} \otimes e_{l}$ for $l \in I_{2}$ and $c_{i} \otimes d_{j}$ can be recovered if each of these copies is available in the surviving nodes. In the discussion below we say that $c_{i} \otimes d_{j}$ is recoverable if either or both of these situations apply.

Thus, it is clear that if $c_{i} \otimes d_{j}$ is not recoverable it has to be the case that all copies of $e_{k^{*}} \otimes d_{j}$ for some $k^{*} \in I_{1}$ are unavailable. This implies that there exists a set of failed nodes denoted $F_{1} \subset \mathcal{N}\left(c_{i}\right) \otimes d_{j}$ of size at least $\rho_{1}-1$. Arguing in a similar vein, we can consider whether $c_{i} \otimes d_{j}$ can be recovered from the nodes in $c_{i} \otimes \mathcal{N}\left(d_{j}\right)$. Based on the discussion above, if $c_{i} \otimes d_{j}$ is not recoverable, it has to be the case that there exists a set of failed nodes $F_{2} \subset c_{i} \otimes \mathcal{N}\left(d_{j}\right)$ of size at least $\rho_{2}-1$. In addition, the node sets $\mathcal{N}\left(c_{i}\right) \otimes d_{j}$ and $c_{i} \otimes \mathcal{N}\left(d_{j}\right)$ are disjoint, thus $F_{1} \cap F_{2}=\emptyset$, i.e., it is clear that there are at least $\rho_{1}+\rho_{2}-2$ failures are essential to ensure that $c_{i} \otimes d_{j}$ is not recoverable.

Next, we examine whether any of the nodes in $F_{1} \cup F_{2}$ are recoverable. A given node in $F_{1}$ is of the form $c_{i^{\prime}} \otimes d_{j}$ where $c_{i}^{t} c_{i^{\prime}}=1$. It is evident that $c_{i^{\prime}} \otimes d_{j}$ cannot be recovered from $\mathcal{N}\left(c_{i^{\prime}}\right) \otimes d_{j}$ as all copies of $e_{k^{*}} \otimes d_{j}$ for a specific $k^{*}$ are unavailable owing to the failure of the nodes in $F_{1}$. Specifically, note that it rules out the possibility of using the
surviving nodes in the set $\mathcal{N}\left(c_{i}\right) \otimes d_{j}$. From the previous observation, it can only be recovered exclusively from the nodes in $c_{i^{\prime}} \otimes \mathcal{N}\left(d_{j}\right)$.

Thus, there need to be at least $\rho_{2}-1$ failures from the node set $c_{i^{\prime}} \otimes \mathcal{N}\left(d_{j}\right)$ to ensure that $c_{i^{\prime}} \otimes d_{j}$ is not recoverable. Furthermore, these failures are distinct from the failures in $F_{1} \cup F_{2}$. Arguing in this way for each node in $F_{1}$, we conclude that at least $\left(\rho_{1}-1\right)\left(\rho_{2}-1\right)$ failures need to be induced to ensure that none of the nodes in $F_{1}$ can be recovered.

However, this implies a total of $1+\rho_{1}+\rho_{2}-2+\left(\rho_{1}-\right.$ 1) $\left(\rho_{2}-1\right)=\rho_{1} \rho_{2}>\rho_{1} \rho_{2}-1$ failures. Thus, we conclude that even if an appropriate $F_{1} \cup F_{2}$ can be found for $c_{i} \otimes d_{j}$, at least one node in $F_{1}$ can be recovered. After this recovery, the set $F_{1}$ cannot exist. This implies that $c_{i} \otimes d_{j}$ can be recovered. As the choice of $c_{i} \otimes d_{j}$ was arbitrary, we can recover any node when there are at most $\rho_{1} \rho_{2}-1$ failures.
This bound is tight since each symbol in $\bar{N}$ is repeated $\rho_{1} \rho_{2}$ times. Thus, we can easily find a set of $\rho_{1} \rho_{2}$ failures that we cannot recover from.

Remark 1: By Skolem's construction [14] the existence of Steiner systems $S(2,3, \theta)$ for all $\theta \geq 7$ and $\theta \equiv 1,3 \bmod 6$ is known. By this fact and Lemma 1, we have an infinite family of FR codes.

Corollary 1: Let $N_{1}$ and $N_{2}$ be transposes of incidence matrices of two Steiner systems with parameters $\left(n_{1}, \theta_{1}, \alpha_{1}, \rho\right)$ and $\left(n_{2}, \theta_{2}, \alpha_{2}, \rho\right)$. Assume that the FR code obtained from $\bar{N}$ has normalized repair bandwidth $\beta=\rho$. Then the FR code is resilient up to $\alpha_{1} \alpha_{2}-1$ failures.

Proof: Any two nodes meet in exactly one symbol in the FR code obtained by transposes of incidence matrices of a Steiner system. Also note that the main ingredient of the proof Lemma 1 is the property that two nodes meet in at most one symbol in Steiner systems. So the rest follows similarly as in the previous proof.
It can be observed that in Example 2, we considered the Kronecker product of a $S(2,2,3)$ system with itself.
Next, we investigate the properties of FR codes that are generated by taking the Kronecker product of resolvable FR codes with themselves.

Lemma 2: Let $N$ be the incidence matrix of resolvable FR code with parameters $(n, \theta, \alpha, \rho)$. Then, the FR code obtained from $\bar{N}=N \otimes N$ is also resolvable.

Proof: We can order the columns of $N$ with respect to the $\rho$ parallel classes. Assume that the $j$-th block in $i$-th parallel class is represented by the column $c_{i, j}$. We will show for fixed $i$ and $s, c_{i, j} \otimes c_{s, r}$ with $1 \leq j \leq \frac{\theta}{\alpha}$ and $1 \leq r \leq \frac{\theta}{\alpha}$ forms a set of blocks which is a parallel class. There will be $\frac{\theta^{2}}{\alpha^{2}}$ blocks in this set, hence it is enough to show that any two distinct blocks do not share any points. Since $\left(c_{i, j}^{t} \otimes c_{s, r}^{t}\right)\left(c_{i, u} \otimes c_{s, v}\right)=$ $\left(c_{i, j}^{t} c_{i, u} \otimes c_{s, r}^{t} c_{s, v}\right)$ equals the zero, the $\frac{\theta^{2}}{\alpha^{2}}$ vectors form a parallel class.

Example 3: A simple example can be obtained from $\mathcal{C}=$ $(\Omega, V)$ where $\Omega=\{1,2,3,4\}$ and $V=\left\{V_{1}=\{1,2\}, V_{2}=\right.$ $\left.\{3,4\}, V_{3}=\{2,3\}, V_{4}=\{1,4\}\right\}$. The code obtained from $N=N \otimes N$ is illustrated in Figure 3.


Fig. 3. The resultant FR code has $\Omega=$ $\{1,2,3,4,5,6,7,8,9, A, B, C, D, E, F, G\}$. Each storage node contains 4 symbols. A failed node can be recovered by contacting two nodes and downloading 2 packets from each. The code is resilient up to 3 failures.

It is a challenging task to determine the exact code rate of a FR code for a given value of $k$. Indeed, determining the code rate of a FR code is equivalent to finding the expansion of $k$-sized node subsets and this in general is known to be a hard problem [15], though in certain cases, the algebraic properties of the constructions can be leveraged to provide a precise estimate; this is topic of ongoing work. However, certain observations can be made.

Lemma 3: Let $\mathcal{C}_{1}=\left(\Omega_{1}, V_{1}\right)$ and $\mathcal{C}_{2}=\left(\Omega_{2}, V_{2}\right)$ be two FR codes with parameters $\left(n_{1}, \theta_{1}, \alpha, \rho_{1}\right)$ and $\left(n_{2}, \theta_{2}, \alpha, \rho_{2}\right)$ such that any two storage nodes in $\mathcal{C}_{1}$ (or $\mathcal{C}_{2}$ ) have at most one symbol in common. Let $\mathcal{M}_{1}$ and $\mathcal{M}_{2}$ denote the file sizes of $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$ respectively for a given $k_{1} \leq \min \left\{n_{1}, n_{2}\right\}$. Suppose either $\mathcal{M}_{1}$ or $\mathcal{M}_{2}$ is equal to $k_{1} \alpha-\binom{k_{1}}{2}$. Then the FR code $\mathcal{C}$ obtained from Kronecker product of $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$ has parameters $\left(n=n_{1} n_{2}, \theta=\theta_{1} \theta_{2}, \alpha^{2}, \rho_{1} \rho_{2}\right)$. The filesize for $\mathcal{C}$ when $k=k_{1}$ is given by $k_{1} \alpha^{2}-\alpha\binom{k_{1}}{2}$.

Proof: Let $N_{1}$ and $N_{2}$ denote the incidence matrices of the FR codes $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$. We know that any two nodes in $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$ have at most one symbol in common. Thus, using a simple inclusion-exclusion principle argument it can be seen that $\mathcal{M}_{i} \geq k_{1} \alpha-\binom{k_{1}}{2}$ for $i=1,2$. Furthermore we are given that one of them meets this lower bound. Without loss of generality we assume that $\mathcal{M}_{1}=k_{1} \alpha-\binom{k_{1}}{2}$. This implies that there exist a set of column vectors $\mathcal{I}_{1}=\left\{c_{1}, \ldots, c_{k_{1}}\right\}$ in $N_{1}$ such that they cover $\mathcal{M}_{1}=k_{1} \alpha-\binom{k_{1}}{2}$ symbols, i.e., any two columns from $\mathcal{I}_{1}$ have exactly one symbol in common and any three columns from $\mathcal{I}_{1}$ have no symbols in common.

First we consider the overlap between any two columns in $N_{1} \otimes N_{2}$. This can be expressed as $\left(c_{i} \otimes d_{j}\right)^{t}\left(c_{i^{\prime}} \otimes d_{j^{\prime}}\right)=$ $c_{i}^{t} c_{i^{\prime}} \otimes d_{j}^{t} d_{j^{\prime}} \leq \alpha$. Thus the overlap between any two columns in $N_{1} \otimes N_{2}$ is at most $\alpha$ and therefore the file size of $\mathcal{C}$ is at least $k_{1} \alpha^{2}-\alpha\binom{k_{1}}{2}$.

Next, we demonstrate a set of columns in $N_{1} \otimes N_{2}$ that meets this lower bound. Let us consider a column in $N_{2}$, denoted $d_{1}$ and examine $N_{1} \otimes d_{1}$. Within, this set we have a subset of $k_{1}$ columns denoted $\mathcal{I}_{2}=\left\{c_{i} \otimes d_{1}\right.$, for $\left.c_{i} \in \mathcal{I}_{1}\right\}$.

Now $\left(c_{i} \otimes d_{1}\right)^{t}\left(c_{j} \otimes d_{1}\right)=c_{i}^{t} c_{j} \otimes d_{1}^{T} d_{1}=\alpha$, whereas any three column vectors from $\mathcal{I}_{2}$ will have a zero overlap. Thus, the number of symbols covered by this set is exactly $k_{1} \alpha^{2}-\alpha\binom{k_{1}}{2}$.

For instance, in Example 3 above, if $k=2$, the individual code has a file size of 3 , whereas the code obtained by taking the Kronecker product has a file size of 6 when $k=2$.

## IV. Conclusions and Future Work

We introduced the Kronecker product technique for the construction of new fractional repetition codes from existing ones. Specifically, we demonstrated that the Kronecker product of two Steiner systems with the same storage capacity $\alpha$ yields a new code with normalized repair bandwidth $\beta=\alpha$ and a failure resilience that is as high as possible. Future work would involve an investigation of more classes of FR codes that yield interesting parameter ranges upon considering the Kronecker product and a study of the corresponding file sizes.

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