An Achievable Region for the Double Unicast Problem Based on a Minimum Cut Analysis

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***Abstract*—We consider the multiple unicast problem under network coding over directed acyclic networks when there are two source-terminal pairs, $s_1 - t_1$ and $s_2 - t_2$. The capacity region for this problem is not known; furthermore, the outer bounds on the region have a large number of inequalities which makes them hard to explicitly evaluate. In this work we consider a related problem. We assume that we only know certain minimum cut values for the network, e.g., mincut(S_i, T_j), where $S_i \subseteq \{s_1, s_2\}$ and $T_j \subseteq \{t_1, t_2\}$ for different subsets S_i and T_j. Based on these values, we propose an achievable rate region for this problem using linear network codes. Towards this end, we begin by defining a multicast region where both sources are multicast to both the terminals. Following this we enlarge the region by appropriately encoding the information at the source nodes, such that terminal t_i is only guaranteed to decode information from the intended source s_i, while decoding a linear function of the other source. The rate region depends upon the relationship of the different cut values in the network.**

***Index Terms*—Network coding, multiple unicast, achievable region.**

I. Introduction

In a multiple unicast connection over a network, there are several source terminal pairs that want to communicate with each other. Each terminal is only interested in receiving messages from its corresponding source. This is in contrast to the multicast problem where each terminal requests exactly the same set of messages from the source nodes. The multicast problem under network coding is very well understood. In particular, several papers [1][2][3] discuss the capacity region and network code construction algorithms for this problem.

However, the multiple unicast problem is not that well understood. A significant amount of previous work has attempted to find inner and outer bounds on the capacity region for a given instance of a multiple unicast network. In [4], an information theoretic characterization for directed acyclic networks is provided. However, this region is not computable as there is no upper bound on the cardinality of the random variables involved in the characterization. The authors in [5] propose an outer bound on the capacity region for general networks. This bound is hard to evaluate even for small sized networks due to the large number of inequalities involved in the characterization. Reference [6] provides an outer bound on the capacity region in a two unicast session network, and presents a network structure in which the outer bound is the exact capacity region. An improved network sharing outer bound was proposed in [7]; it was shown to be the tightest bound that can be realized with edge-cut bounds. The work of [8] proposes an achievable scheme by considering butterfly structures along with XOR coding in the network. Similarly, the work of [9] presents a rate region that can be supported by XOR coding between pairs of flows. Multiple unicast has been studied in [10], [11] for networks with link faults and errors; however, the topologies of these networks are restricted (though realistic in the protection context).

Several papers have focused on the case of two unicast networks. For instance, the work of [12] (see also [13]) presented a necessary and sufficient condition on the network structure for the existence of a network coding solution that supports unit rate transmission for each $s_i - t_i$ connection. Reference [14] considered directed acyclic networks and proposed an achievable rate region for this problem based on the number of edge disjoint paths for each $s_i - t_i$ connection.

More recent work has considered networks with three unicast sessions. Work by the present authors [15], [16], [17] considered unit rate transmission in such networks and references [18][19][20] discuss the usage of interference alignment in the network coding context.

In this work we propose an achievable region for the two-unicast problem using linear network codes. We consider directed acyclic networks with unit capacity edges and assume that we only know certain minimum cut values for the network, e.g., mincut(S_i, T_j), where $S_i \subseteq \{s_1, s_2\}$ and $T_j \subseteq \{t_1, t_2\}$ for different subsets S_i and T_j. We classify networks according to the relationship of the different cut values of the network. To find the achievable region, we first find a multicast region where both sources can be multicast to the terminals. Subsequently, this region is extended according to the specific class that the network belongs to. Our achievability scheme uses random linear network coding and appropriate precoding at the sources. Following the publication of our preliminary conference paper [21] (and the submission of the present manuscript), certain results have appeared in the literature that we now ouline. The work of [22] derives an achievable rate region by treating the two unicast problem as an instance of a two-user linear deterministic interference channel. Reference [22] uses the Han-Kobayashi scheme, i.e., splits the messages into private and common parts and arrives at an achievable region that is larger than our proposed region. The authors in [23] also derive an achievable rate region in terms of the cut values. For some networks, our scheme achieves a larger region than theirs.

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This paper is organized as follows. Section II introduces the system model under consideration. Sections III and IV contain the precise problem formulation and the derivations of our proposed achievable rate region according to the different cut values. Section V compares our achievable region to existing literature and Section VI concludes this paper.

II. System Model

We consider a network represented by a directed acyclic graph $G = (V, E)$. There is a source set $S = \{s_1, s_2\} \in V$ in which each source observes a random process (the processes are independent) with a discrete integer entropy, and there is a terminal set $T = \{t_1, t_2\} \in V$ in which t_i needs to uniquely recover the information transmitted from s_i at rate R_i. Each edge $e \in E$ has unit capacity and can transmit one symbol from a finite field of size q. If a given edge has a higher capacity, it can be divided into multiple parallel edges with unit capacity. Without loss of generality (W.l.o.g.), we assume that there is no incoming edge into source s_i, and no outgoing edge from terminal t_i. By Menger's theorem, the minimum cut between sets $S_{N_1} \subseteq S$ and $T_{N_2} \subseteq T$ is the number of edge disjoint paths from S_{N_1} to T_{N_2}, and will be denoted by $k_{N_1-N_2}$ where $N_1, N_2 \subseteq \{1, 2\}$. For two unicast sessions, we define the *cut vector* as the vector of the cut values k_{1-1}, k_{2-2}, k_{1-2}, k_{2-1}, k_{12-1}, k_{12-2}, k_{1-12}, k_{2-12} and k_{12-12}.

The network coding model in this work is based on [2]. Assume that source s_i needs to transmit at rate R_i. Then the random variable observed at s_i is denoted as $X_i = (X_{i1}, X_{i2}, \cdots, X_{iR_i})$, where each X_{ij} is an element of the finite field of size q denoted by $GF(q)$. For linear network codes, the signal on an edge (i, j) is a linear combination of the signals on the incoming edges on i or a linear combination of the source signals at i. Let Y_{e_n} ($tail(e_n) = k$ and $head(e_n) = l$) denote the signal on edge $e_n \in E$. Then, we have

$$Y_{e_n} = \sum_{\{e_m | head(e_m)=k\}} f_{m,n} Y_{e_m} \quad \text{if } k \in V \setminus \{s_1, s_2\}, \text{ and}$$

$$Y_{e_n} = \sum_{j=1}^{R_i} a_{ij,n} X_{ij} \quad \text{if } X_i \text{ is observed at } k.$$

The local coding vectors $a_{ij,n}$ and $f_{m,n}$ are also chosen from $GF(q)$. We can also express Y_{e_n} as $Y_{e_n} = \sum_{j=1}^{R_1} \alpha_{j,n} X_{1j} + \sum_{j=1}^{R_2} \beta_{j,n} X_{2j}$. The global coding vector of Y_{e_n} is $[\alpha_n, \beta_n] = [\alpha_{1,n}, \cdots, \alpha_{R_1,n}, \beta_{1,n}, \cdots, \beta_{R_2,n}]$. We are free to choose an appropriate value of the field size q.

In this work, we present an achievable rate region given the cut vector; namely, k_{1-1}, k_{2-2}, k_{1-2}, k_{2-1}, k_{12-1}, k_{12-2}, k_{1-12}, k_{2-12} and k_{12-12}. W.l.o.g, we assume that there are k_{i-ij} outgoing edges from s_i and k_{ij-i} incoming edges into t_i. If this is not the case one can always introduce an artificial source (terminal) node connected to the original source (terminal) node by k_{i-ij} (k_{ij-i}) edges. It can be seen that the new network has the same cut vector as the original network.

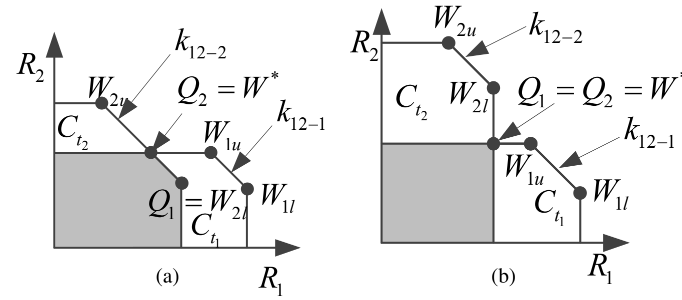

Fig. 1. (a) An example of C_{t_1} and C_{t_2} when the multicast region shaded is pentagonal. (b) Another example where the multicast region is rectangular.

III. Achievable Rate Region for Given $k_{12-1}, k_{12-2}, k_{1-1}, k_{2-2}, k_{1-2}$, and k_{2-1}

We first consider the case that a subset of the cut values in the cut vector are available, namely, $k_{12-1}, k_{12-2}, k_{1-1}, k_{2-2}, k_{1-2}$, and k_{2-1}. Suppose for now that only t_1 is interested in recovering both the random variables X_1 and X_2 which are observed at s_1 and s_2 respectively. Denote the rate from s_1 to t_1 and s_2 to t_1 as R_{11} and R_{12}. The rate pairs (R_{11}, R_{12}) are achieved via routing [24] and the corresponding capacity region C_{t_1} is given by

$$C_{t_1} = \{R_{11} \le k_{1-1}, \;\; R_{12} \le k_{2-1}, \;\; R_{11} + R_{12} \le k_{12-1}\}.$$

The capacity region C_{t_2} for t_2 can be drawn in a similar manner (an example is shown in Fig. 1(a)). We also find the boundary points $W_{1u}, W_{1l}, W_{2u}, W_{2l}$[1] such that their coordinates are $W_{1u} = (k_{12-1} - k_{2-1}, k_{2-1}), W_{1l} = (k_{1-1}, k_{12-1} - k_{1-1}), W_{2u} = (k_{12-2} - k_{2-2}, k_{2-2}), W_{2l} = (k_{1-2}, k_{12-2} - k_{1-2})$. A simple achievable rate region for our problem can be arrived at by multicasting both sources X_1 and X_2 to both the terminals t_1 and t_2.

Lemma 3.1: Rate pairs (R_1, R_2) belonging to the following set $\mathcal{B}$ can be achieved for two unicast sessions.

$$\begin{aligned}\mathcal{B} = \{&R_1 \le \min(k_{1-2}, k_{1-1}), R_2 \le \min(k_{2-1}, k_{2-2}),\\ &R_1 + R_2 \le \min(k_{12-1}, k_{12-2})\}.\end{aligned}$$

Proof: We multicast both the sources to each terminal. This can be done using the multi-source multi-sink multicast result (Thm. 8 in [2]). ■

Subsequently we will refer to region $\mathcal{B}$ achieved by multicast as the *multicast region* (the grey region in Fig. 1(a)). It can be observed that if the cut values are such that

$$\min(k_{1-2}, k_{1-1}) + \min(k_{2-1}, k_{2-2}) \le \min(k_{12-1}, k_{12-2}), \tag{1}$$

then the region is rectangular (Fig. 1(b)), otherwise, it is pentagonal (Fig. 1(a)).

We now move on to precisely formulating the problem. Let Z_i denote the received vector at t_i, X_i denote the transmitted vector at s_i, and H_{ij} denote the transfer function from s_j to t_i. Let M_i denote the encoding matrix at s_i, i.e., M_i is the transformation from X_i to the transmitted symbols on the outgoing edges from s_i. In our formulation, we will let the length of X_i to be k_{i-i}, i.e., the maximum possible. For transmission at rates R_1 and R_2, we introduce precoding

[1] subscripts l and u are meant to denote lower and upper.

TABLE I
DIMENSION AND RANK OF MATRICES

matrix	*dimension*	*rank*
H_{11}	$k_{12-1} \times k_{1-12}$	k_{1-1}
H_{12}	$k_{12-1} \times k_{2-12}$	k_{2-1}
$[H_{11} \;\; H_{12}]$	$k_{12-1} \times (k_{1-12} + k_{2-12})$	k_{12-1}
H_{21}	$k_{12-2} \times k_{1-12}$	k_{1-2}
H_{22}	$k_{12-2} \times k_{2-12}$	k_{2-2}
$[H_{21} \;\; H_{22}]$	$k_{12-2} \times (k_{1-12} + k_{2-12})$	k_{12-2}

matrices $V_i, i = 1, 2$ of dimension $R_i \times k_{i-i}$, so that the overall system of equations is as follows.

$$\begin{aligned} Z_1 &= H_{11} M_1 V_1 X_1 + H_{12} M_2 V_2 X_2, \\ Z_2 &= H_{21} M_1 V_1 X_1 + H_{22} M_2 V_2 X_2. \end{aligned} \tag{2}$$

We say that t_i can receive information at rate R_i from s_i if it can decode $V_i X_i$ perfectly; each entry in V_i is either 0 or 1. The row dimension of the V_i's can be adjusted to obtain different rate vectors. Under random linear network coding, it can be shown that there exist local coding vectors over a large enough field such that the ranks of the different matrices in the first column of Table I are given by the corresponding entries in the third column, which correspond to the maximum possible. Furthermore, by the multi-source multi-sink multicast result [2], when $(R_1, R_2) \in \mathcal{B}$ these matrices are such that $[H_{11} M_1 \;\; H_{12} M_2]$ is a full column rank matrix of dimension $k_{12-1} \times (R_1 + R_2)$, and $[H_{21} M_1 \;\; H_{22} M_2]$ is a full column rank matrix of dimension $k_{12-2} \times (R_1 + R_2)$. In Table I, for instance since the minimum cut between s_1 and t_1 is k_{1-1}, we know that the maximum rank of H_{11} is k_{1-1}. Using the formalism of [2], we can conclude that there is a square submatrix of H_{11} of dimension $k_{1-1} \times k_{1-1}$ whose determinant is not identically zero. Such appropriate submatrices can be found for each of the matrices in the first column of Table I. This in turn implies that their product is not identically zero and therefore using the Schwartz-Zippel lemma [25], we can conclude that there exists an assignment of local coding vectors over a sufficiently large finite field so that the rank of all the matrices is simultaneously the maximum possible. While, the Schwartz-Zippel lemma requires random choice of the local coding vectors, the probability of success in the algorithm can be made arbitrarily close to one if the field size is chosen large enough, or through repeated trials, hence it runs in random polynomial time. For the rest of the paper, we assume that such a choice of local coding vectors has been made. Our arguments will revolve around appropriately modifying source encoding matrices M_1 and M_2.

Note that in general the multicast region has a pentagonal shape (see Fig. 1(a)). Two points on this pentagon (denoted as Q_1 and Q_2) are of specific interest. At point Q_1, we denote the achievable rate pair by (R_1^*, R_2^*) where

$$\begin{aligned} &R_1^* = \min(k_{1-2}, k_{1-1}), \text{ and} \\ &R_2^* = \min(\min(k_{2-1}, k_{2-2}), \min(k_{12-1}, k_{12-2}) - R_1^*). \end{aligned}$$

If the region is pentagonal, then $R_1^* = \min(k_{1-2}, k_{1-1})$ and $R_2^* = \min(k_{12-1}, k_{12-2}) - R_1^*$. Likewise at point Q_2, we denote the achievable rate pair by (R_1^{**}, R_2^{**}) where

$$\begin{aligned} &R_1^{**} = \min(\min(k_{1-2}, k_{1-1}), \min(k_{12-1}, k_{12-2}) - R_2^{**}), \text{ and} \\ &R_2^{**} = \min(k_{2-1}, k_{2-2}). \end{aligned}$$

If the region is pentagonal, then $R_1^{**} = \min(k_{12-1}, k_{12-2}) - R_2^{**}$ and $R_2^{**} = \min(k_{2-1}, k_{2-2})$. If the region is rectangular, then $Q_1 = Q_2$, and $R_1^* = R_1^{**} = \min(k_{1-2}, k_{1-1})$ and $R_2^* = R_2^{**} = \min(k_{2-1}, k_{2-2})$. In Fig. 1(a), these boundary points are $Q_1 = W_{2l}$ and $Q_2 = W^*$, and the multicast region is pentagonal. Another example is shown in Fig. 1(b) where $Q_1 = Q_2$ and the multicast region is rectangular.

In what follows, we will present our arguments towards increasing the value of R_1 and R_2 to achieve points that are near Q_1 but do not belong to $\mathcal{B}$. In this paper we refer to $k_{1-2} + k_{2-1}$ as a measure of the interference in the network and in the subsequent discussion present achievable regions based on its value. We emphasize though that this is nomenclature used for ease of presentation. Indeed a high value of k_{1-2} does not necessarily imply that there is a lot of interference at t_2, since the network code itself dictates the amount of interference seen by t_2. The following lemma will be used extensively.

Lemma 3.2: Consider a system of equations $Z = H_1 X_1 + H_2 X_2$, where X_1 is a vector of length l_1 and X_2 is a vector of length l_2 and $Z \in span([H_1 \;\; H_2])^2$. The matrix H_1 has dimension $z_t \times l_1$, and rank $l_1 - \sigma$, where $0 \leq \sigma \leq l_1$. The matrix H_2 is full rank and has dimension $z_t \times l_2$ where $z_t \geq (l_1 + l_2 - \sigma)$. Furthermore, the column spans of H_1 and H_2 intersect only in the all-zeros vectors, i.e., $span(H_1) \cap span(H_2) = \{0\}$. Then, there exists a unique solution for X_2.

Proof: As $Z \in span([H_1 \;\; H_2])$, there exist X_1 and X_2 such that $Z = H_1 X_1 + H_2 X_2$. Now assume that there exist X_1' and X_2' (different from X_1 and X_2) such that $Z = H_1 X_1' + H_2 X_2'$. This implies

$$H_1(X_1 - X_1') = H_2(X_2 - X_2'). \tag{3}$$

Since $span(H_1) \cap span(H_2) = \{0\}$, both sides of eq. (3) are zero. Furthermore, since H_2 is a full rank matrix, $X_2 = X_2'$, i.e., the solution for X_2 is unique. ■

We next define the achievable rate region which will be used in the rest of the paper.

Definition 3.3: A rate point (R_1, R_2) is said to lie in the *achievable rate region* $\mathcal{R}_A$ if there exist full column rank source encoding matrices M_1 and M_2 where $rank(M_1) = R_1$ and $rank(M_2) = R_2$ such that

$$\begin{aligned} &rank(H_{11} M_1) = rank(M_1), \; rank(H_{22} M_2) = rank(M_2), \\ &span(H_{i1} M_1) \cap span(H_{i2} M_2) = \{0\} \text{ for } i = 1, 2. \end{aligned} \tag{4}$$

The condition above will be referred in the remainder of the paper as the achievable condition. It can be observed that the multicast region $\mathcal{B}$ is a subset of $\mathcal{R}_A$.

[2]Throughout the paper, $span(A)$ refers to the column span of A.

A. Low Interference Case - $k_{1-2}+k_{2-1} \leq \min(k_{12-1}, k_{12-2})$

Note that it always holds that $k_{2-1} + k_{1-1} \geq k_{12-1}$ and $k_{1-2} + k_{2-2} \geq k_{12-2}$. Together with the low interference condition, this implies that $k_{1-1} \geq k_{1-2}$ and $k_{2-2} \geq k_{2-1}$. It follows that the multicast region is a rectangle since eq. (1) is satisfied and $R_1^* = k_{1-2}, R_2^* = k_{2-1}$. Furthermore, $Q_1 = Q_2 = W^*$ as shown in the example in Fig. 1(b).

Our solution strategy is to first consider the encoding matrices M_1 and M_2 at the point Q_1, and to introduce a new encoding matrix at s_1, denoted M_1' (with $R_1^* + \delta$ columns) such that $span(H_{11}M_1') \cap span(H_{12}) = \{0\}$. As shown below, this will allow t_1 to decode from s_1 at rate $R_1^* + \delta$ and t_2 to decode from s_2 at rate R_2^*. After the modification, each t_i is guaranteed to decode at the appropriate rate from s_i. A similar argument applies for R_2^* to arrive at the achievable rate region. At the point Q_1, as both terminals can decode both sources, it holds that

$$rank(H_{i1}M_1) = k_{1-2}, rank(H_{i2}M_2) = k_{2-1}, \text{ and}$$
$$span(H_{i1}M_1) \cap span(H_{i2}M_2) = \{0\} \text{ for } i = 1, 2.$$

Before stating the main result, we present the following lemma.

Lemma 3.4: Rate Increase Lemma. Consider a rate point $(R_1, R_2) \in \mathcal{R}_A$ with corresponding matrices M_1 and M_2 such that (1) $rank([H_{11} \quad H_{12}M_2]) = r > rank([H_{11}M_1 \quad H_{12}M_2]) = R_1 + \Delta$, where $rank(H_{12}M_2) = \Delta \leq R_2$ and (2) $rank([H_{21}M_1]) = rank(H_{21})$. There exist matrices M_1' and M_2' such that t_1 can decode at rate $r - \Delta$ and t_2 can decode at rate R_2.

Proof: We first prove that if M_1 and M_2 satisfy Condition (1), then there exist a series of full rank matrices $\bar{M}_1^{(n)} = [\tilde{M}_1^{(n)} \quad M_1]$ of dimension $k_{1-12} \times (n + R_1)$ such that $rank([H_{11}\bar{M}_1^{(n)} \quad H_{12}M_2]) = R_1 + \Delta + n$, $0 \leq n \leq (r - R_1 - \Delta)$. We prove this part by induction. When $n = 0$, $\bar{M}_1^{(0)} = M_1$, $rank([H_{11}\bar{M}_1^{(0)} \quad H_{12}M_2]) = R_1 + \Delta$.

Assume that when $n = l \leq r - 1 - R_1 - \Delta$, $\bar{M}_1^{(n)}$ can be found such that $rank([H_{11}\bar{M}_1^{(l)} \quad H_{12}M_2]) = R_1 + \Delta + l$. When $n = l + 1 \leq r - R_1 - \Delta$, if there does not exist an $\bar{M}_1^{(l+1)}$, all the columns in $[H_{11} \quad H_{12}M_2]$ are linear combinations of $[H_{11}\bar{M}_1^{(l)} \quad H_{12}M_2]$, which contradicts the fact that $rank([H_{11} \quad H_{12}M_2]) = r > r - 1 \geq l + R_1 + \Delta$. Hence, there must exist a series of full rank matrices $\bar{M}_1^{(n)}$ such that $rank([H_{11}\bar{M}_1^{(n)} \quad H_{12}M_2]) = R_1 + \Delta + n$ is satisfied when $0 \leq n \leq r - R_1 - \Delta$.

Next, we prove that t_1 can decode at rate $r - \Delta$ and t_2 can decode at rate R_2 using $M_1' = \bar{M}_1^{(r-R_1-\Delta)}$ and $M_2' = M_2$.

Decoding at t_1: Since M_1' is a full rank matrix of dimension $k_{1-12} \times (r - \Delta)$, it also satisfies (i) $rank(H_{11}M_1') = r - \Delta$ and (ii) $span(H_{11}M_1') \cap span(H_{12}M_2) = \{0\}$ because of the following argument. We have

$$\begin{aligned} r &= rank([H_{11}M_1' \quad H_{12}M_2]) \\ &\leq rank([H_{11}M_1']) + rank([H_{12}M_2]) \\ &\leq rank(M_1') + rank(H_{12}M_2) = r - \Delta + \Delta = r. \end{aligned}$$

Then all the inequalities become equalities and (i) and (ii) are satisfied. Then by Lemma 3.2 and the above conditions, t_1 can decode at rate $r - \Delta$.

Decoding at t_2: From Condition (2), we have $span(H_{21}M_1) = span(H_{21})$ (see Lemma A.1 in the Appendix). Furthermore, since $span(M_1) \subseteq span(M_1')$, we have $span(H_{21}M_1) \subseteq span(H_{21}M_1') \subseteq span(H_{21})$. This implies that $span(H_{21}M_1) = span(H_{21}M_1') = span(H_{21})$. Furthermore, since $span(H_{21}M_1) \cap span(H_{22}M_2) = \{0\}$, we also have $span(H_{21}M_1') \cap span(H_{22}M_2) = \{0\}$. Then by Lemma 3.2 and the fact that $H_{22}M_2$ is full rank, t_2 can decode at rate R_2. ∎

Lemma 3.5: If $k_{1-2} + k_{2-1} \leq \min(k_{12-1}, k_{12-2})$, the rate pair in the following region can be achieved.

$$R_1 \leq k_{12-1} - k_{2-1}, \qquad R_2 \leq k_{12-2} - k_{1-2}.$$

Proof: In this case, $(R_1^*, R_2^*) = (k_{1-2}, k_{2-1})$ is the boundary point $Q_1 = Q_2$. Let M_1 and M_2 denote the source encoding matrices at Q_1.

First, note that $rank(H_{12}M_2) = rank(H_{12}) = k_{2-1}$, which implies that $span(H_{12}) = span(H_{12}M_2)$. Therefore

$$\begin{aligned} rank([H_{11} \quad H_{12}]) &= rank([H_{11} \quad H_{12} \quad H_{12}M_2] \\ &= rank([H_{11} \quad H_{12}M_2]) \end{aligned}$$

This implies that $rank([H_{11} \quad H_{12}M_2]) = k_{12-1} \geq k_{1-2} + k_{2-1} = rank([H_{11}M_1 \quad H_{12}M_2])$ since by assumption $k_{1-2} + k_{2-1} \leq \min(k_{12-1}, k_{12-2})$. Moreover, $rank(H_{21}M_1) = rank(H_{21}) = k_{1-2}$. Therefore by the Rate Increase Lemma, we can achieve rate point $(R_1 = k_{12-1} - k_{2-1}, R_2 = k_{2-1})$. Using a similar argument, we can further increase R_2 such that rate pair $(k_{12-1} - k_{2-1}, k_{12-2} - k_{1-2})$ can be achieved. This region is the hatched gray region in Fig. 2. ∎

This implies that the point $W' = (k_{12-1} - k_{2-1}, k_{12-2} - k_{1-2})$ is achievable. Also note that since we applied the Rate Increase Lemma, we have $rank([H_{11}M_1' \quad H_{12}M_2]) = rank([H_{11} \quad H_{12}M_2])$. Next, we consider the scenario in which rates can be traded off between the two unicast sessions.

Lemma 3.6: Rate Exchange Lemma – 1-1 tradeoff. Consider a rate point $(R_1, R_2) \in \mathcal{R}_A$ with corresponding matrices M_1 and M_2.

(a) If M_1 and M_2 satisfy (1) $rank([H_{11}M_1 \quad H_{12}M_2]) = rank([H_{11} \quad H_{12}M_2]) = r$, where $R_1 + R_2 \geq r$, and (2) $rank(H_{21}M_1) = rank(H_{21})$, there exist M_1' and M_2' such that t_1 can decode at rate $\min(R_1 + 1, k_{1-1})$ and t_2 can decode at rate $\max(R_2 - 1, 0)$.

(b) If M_1 and M_2 satisfy (1) $rank([H_{11} \quad H_{12}M_2]) = r > rank([H_{11}M_1 \quad H_{12}M_2]) = R_1 + \Delta$, where $rank(H_{12}M_2) = \Delta \leq R_2$, and (2) $rank(H_{21}M_1) < rank(H_{21})$, there exist M_1' and M_2' such that t_1 can decode at rate $\min(R_1 + 1, k_{1-1})$ and t_2 can decode at rate $\max(R_2 - 1, 0)$.

Lemma 3.7: Rate Exchange Lemma – 1-2 tradeoff. Consider a rate point $(R_1, R_2) \in \mathcal{R}_A$ with corresponding matrices M_1 and M_2. If M_1 and M_2 satisfy (1) $rank([H_{11}M_1 \quad H_{12}M_2]) = rank([H_{11} \quad H_{12}M_2]) = r$, where $R_1 + R_2 \geq r$, and (2) $rank(H_{21}M_1) < rank(H_{21})$, there exist M_1'' and M_2'' such that t_1 can decode at rate $\min(R_1 + 1, k_{1-1})$ and t_2 can decode at rate $\max(R_2 - 2, 0)$.

Proof: 1-1 tradeoff. We assume that $R_1 + 1 \leq k_{1-1}$ and $R_2 - 1 \geq 0$. A vector $\vec{\alpha}$ is added to M_1 to form M_1' such that

$M_1' = [\vec{\alpha} \quad M_1]$ and $rank(H_{11}M_1') = R_1 + 1$ where $H_{11}M_1'$ is of dimension $k_{12-1} \times (R_1 + 1)$.

For part (a), because of Condition (1), $H_{11}\vec{\alpha}$ will be a nonzero linear combination of the vectors in $H_{11}M_1$ and $H_{12}M_2$, i.e., $H_{11}\vec{\alpha} = H_{11}M_1\vec{\gamma}_1 + H_{12}M_2\vec{\gamma}_2$. Note that $\vec{\gamma}_1$ is unique; otherwise, assume that there exist $\vec{\gamma}_1'$ and $\vec{\gamma}_2'$ such that $H_{11}\vec{\alpha} = H_{11}M_1\vec{\gamma}_1' + H_{12}M_2\vec{\gamma}_2'$ where $\vec{\gamma}_1' \neq \vec{\gamma}_1$. If $H_{12}M_2\vec{\gamma}_2 = H_{12}M_2\vec{\gamma}_2'$ then $H_{11}M_1\vec{\gamma}_1 = H_{11}M_1\vec{\gamma}_1'$ which indicates that $H_{11}M_1$ is not full column rank. On the other hand if $H_{12}M_2\vec{\gamma}_2 \neq H_{12}M_2\vec{\gamma}_2'$, then it means that $span(H_{11}M_1) \cap span(H_{12}M_2) \neq \{0\}$. Hence, by contradiction, we have $\vec{\gamma}_1' = \vec{\gamma}_1$, which indicates that $\vec{\gamma}_1$ is unique. Then, $\vec{\beta} = H_{11}\vec{\alpha} - H_{11}M_1\vec{\gamma}_1$ is a vector which contains at least one nonzero element. Otherwise, if $\vec{\beta}$ is a zero vector, $rank(H_{11}M_1')$ will be rank R_1 which is a contradiction. Assume w.l.o.g. that the nonzero element is on the first row of $\vec{\beta}$.

Next, we select a full rank matrix U of dimension $R_2 \times (R_2 - 1)$ from the null space of the first row of $H_{12}M_2$ such that the first row of $H_{12}M_2U$ is a zero row vector. It follows that $H_{11}\vec{\alpha}$ can not be represented by a linear combination of the vectors in $H_{11}M_1$ and $H_{12}M_2U$, which indicates that $H_{11}\vec{\alpha} \notin span([H_{11}M_1 \quad H_{12}M_2U])$. Next, because $span(H_{11}M_1) \cap span(H_{12}M_2) = \{0\}$, we have $span(H_{11}M_1) \cap span(H_{12}M_2U) = \{0\}$. Finally, we conclude that $span(H_{11}M_1') \cap span(H_{12}M_2') = \{0\}$ where $M_2' = M_2U$. Hence, t_1 can decode at rate $\min(R_1 + 1, k_{1-1})$.

For part (a) if Condition (2) is satisfied, $span(H_{21}M_1) = span(H_{21})$. Using an argument similar to the one used in the proof of Lemma 3.4, it can be shown that $span(H_{21}M_1') = span(H_{21}) = span(H_{21}M_1)$. This implies that $span(H_{21}M_1') \cap span(H_{22}M_2') = \{0\}$ since $span(H_{22}M_2') \subseteq span(H_{22}M_2)$. Then t_2 can decode at rate $R_2 - 1$ since $rank(H_{22}M_2') = R_2 - 1$.

For part (b) if Condition (1) is satisfied, we can find an M_1' such that $rank(H_{11}M_1') = R_1 + 1$ and $span(H_{11}M_1') \cap span(H_{12}M_2) = \{0\}$. At the same time, if Condition (2) of part (b) is satisfied, $rank(H_{21}M_1') - rank(H_{21}M_1) \leq 1$. Then $rank(span(H_{21}M_1') \cap span(H_{22}M_2))$ can be as large as 1. As $H_{22}M_2$ is a full column rank matrix, we can find an M_2' by deleting one column from M_2 such that $span(H_{21}M_1') \cap span(H_{22}M_2') = \{0\}$ where M_2' is a full rank matrix of dimension $k_{2-12} \times (R_2 - 1)$. Furthermore, since $span(H_{12}M_2') \subseteq span(H_{12}M_2)$, we will have that $span(H_{11}M_1') \cap span(H_{12}M_2') = \{0\}$. With this M_1' and M_2', the rate point $(R_1 + 1, R_2 - 1)$ can be achieved. ■

Proof: 1-2 tradeoff. We assume that $R_1 + 1 \leq k_{1-1}$ and $R_2 - 2 \geq 0$.

Note that Condition (1) here is the same as in the Rate Exchange Lemma – 1-1 tradeoff – part(a). Therefore, we can find two matrices M_1' and M_2' with rank $R_1 + 1$ and $R_2 - 1$ by appending one vector to M_1 and selecting $M_2' = M_2U$ such that $rank(H_{11}M_1') = R_1 + 1$, and $span(H_{11}M_1') \cap span(H_{12}M_2') = \{0\}$ where U is a full rank matrix of dimension $R_2 \times (R_2 - 1)$ such that $rank(H_{12}M_2) - rank(H_{12}M_2U) = 1$.

If Condition (2) is satisfied, $rank(H_{21}M_1') - rank(H_{21}M_1)$ can be as large as 1. Then $rank(span(H_{21}M_1') \cap span(H_{22}M_2'))$ can be as large as 1. Because $H_{22}M_2'$ is a full column rank matrix, we can find an M_2'' by deleting one column from M_2' such that $span(H_{21}M_1') \cap span(H_{22}M_2'') = \{0\}$ where M_2'' is a full rank matrix of dimension $k_{2-12} \times (R_2 - 2)$. Furthermore, since $span(H_{12}M_2'') \subseteq span(H_{12}M_2')$, we will have that $span(H_{11}M_1') \cap span(H_{12}M_2'') = \{0\}$. Finally let $M_1'' = M_1'$. With encoding matrices M_1'' and M_2'', it can be seen that $(R_1 + 1, R_2 - 2)$ can be achieved. ■

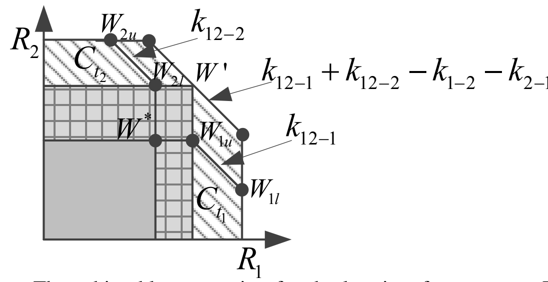

Fig. 2. The achievable rate region for the low interference case. For each point in the shaded grey area, both terminals can recover both the sources. In the hatched grey area and the hatched white area, for a given rate point, its x-coordinate is the rate for $s_1 - t_1$ and its y-coordinate is the rate for $s_2 - t_2$; the terminals are not guaranteed to decode both sources in this region. The union of the hatched white region, the hatched gray region and the gray region is the final extended rate region for the low interference case.

By applying the Rate Exchange Lemma – 1-1 tradeoff – part (a), at point $W' = (k_{12-1} - k_{2-1}, k_{12-2} - k_{1-2})$, we have the following theorem.

Theorem 3.8: If $k_{1-2} + k_{2-1} \leq \min(k_{12-1}, k_{12-2})$, the following rate region (see Fig. 2) can be achieved.

Region 1:

$$R_1 \leq k_{1-1}, \qquad R_2 \leq k_{2-2},$$
$$R_1 + R_2 \leq k_{12-1} - k_{2-1} + k_{12-2} - k_{1-2}.$$

Proof: Note that point $W' = (R_1, R_2) = (k_{12-1} - k_{2-1}, k_{12-2} - k_{1-2})$ is achieved by using the Rate Increase Lemma. Let M_1 and M_2 be the encoding matrices at W'. Then, we have $rank([H_{11}M_1 \quad H_{12}M_2]) = rank([H_{11} \quad H_{12}M_2])$, and we further have that $rank(H_{21}M_1) = rank(H_{21}) = k_{1-2}$. Applying the Rate Exchange Lemma – 1-1 tradeoff – part (a) we have the required conclusion. ■

Remark 3.9: Note that it always holds that $k_{12-1} \geq k_{1-1}$, $k_{12-2} \geq k_{2-2}$. Along with the low interference condition, we can conclude that $k_{12-1} - k_{2-1} + k_{12-2} - k_{1-2} \geq \max(k_{1-1}, k_{2-2}) \geq (k_{1-1} + k_{2-2})/2$. As $k_{1-1} + k_{2-2}$ is always an upper bound (albeit loose) on $R_1 + R_2$, this implies that our rate region is within a multiplicative gap of two of the outer bound.

B. High Interference Case- $k_{1-2} + k_{2-1} > \min(k_{12-1}, k_{12-2})$

Note that for the low interference case, the low interference condition implies that $k_{1-1} \geq k_{1-2}$ and $k_{2-2} \geq k_{2-1}$. However, in high interference case, there are several possibilities. We show a case where $k_{1-1} \leq k_{1-2}$ and $k_{2-2} \leq k_{2-1}$ in Fig. 3(a). When $k_{1-1} \geq k_{1-2}$, Fig. 3(b) illustrates an example where $k_{2-2} \leq k_{2-1}$, and Fig. 1(a) (in Section III-A) illustrates an example where $k_{2-2} \geq k_{2-1}$. It can be observed here that unlike the low interference case, Q_1 may not be the same point as Q_2. In the discussion below we present rate regions by extending them from the rate points Q_1 and Q_2.

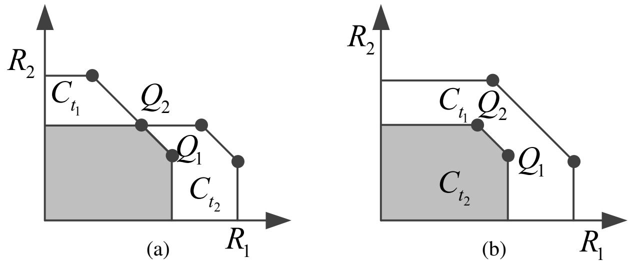

Fig. 3. (a) High interference case where $k_{1-1} \leq k_{1-2}$ and $k_{2-2} \leq k_{2-1}$. (b) High interference case where $k_{1-1} \geq k_{1-2}$ and $k_{2-2} \leq k_{2-1}$.

Claim 3.10: When $Q_1 \neq Q_2$, the Rate Increase Lemma cannot be applied to increase the rate to t_2 above R_2^* at Q_1 or to increase the rate to t_1 above R_1^{**} at Q_2.

Proof: As $Q_1 \neq Q_2$, using eq. (1), we conclude that $\min(k_{1-2}, k_{1-1}) + \min(k_{2-1}, k_{2-2}) > \min(k_{12-1}, k_{12-2})$. Then at Q_1, $R_2^* = \min(\min(k_{2-1}, k_{2-2}), \min(k_{12-1}, k_{12-2}) - \min(k_{1-2}, k_{1-1})) < \min(k_{2-1}, k_{2-2}) \leq k_{2-1}$. Next, since $rank(H_{12}M_2) \leq rank(M_2) = R_2^* < rank(H_{12}) = k_{2-1}$, Condition (2) of the Rate Increase Lemma is not satisfied. A similar argument applies for Q_2. ■

In view of the above claim, using our achievable strategies one can at best use the Rate Exchange Lemma to increase the rate to t_2 at Q_1 while reducing the rate to t_1. As $Q_1 \neq Q_2$, the multicast region is a pentagon and applying the 1-1 tradeoff will at most allow us to achieve the boundary between Q_1 and Q_2, while the 1-2 tradeoff achieves interior points in the multicast region. As points on the $Q_1 - Q_2$ boundary are already achieved by multicasting both sources, the region is not enlarged.

Hence, we will consider rate points (R_1, R_2) such that $R_1 > R_1^*$ and $R_2 = R_2^*$ at Q_1 (and similarly $R_1 = R_1^{**}$ and $R_2 > R_2^{**}$ at Q_2). At Q_1, if $k_{1-2} \geq k_{1-1}$, $R_1^* = k_{1-1}$, i.e. increasing R_1 is impossible since it attains its maximum. Therefore, we assume that $k_{1-2} < k_{1-1}$. By the high interference condition and the fact that $k_{1-2} + k_{2-2} \geq k_{12-2}$, we have $(R_1^*, R_2^*) = (k_{1-2}, \min(k_{12-1}, k_{12-2}) - k_{1-2})$. We begin by modifying the source encoding matrices at point Q_1, with the goal of increasing R_1 the rate to t_1 above R_1^*. Our strategy at Q_1 is similar to the one for the low interference case, namely, we attempt to trace a region of achievable rates by using the Rate Increase and Rate Exchange lemmas. The main difference is that here we also use the 1-2 tradeoff result (cf. Lemma 3.7). Note that in the discussion below, we present the arguments for increasing rates at Q_1 and Q_2 separately. However, if $Q_1 = Q_2$, then the arguments are still applicable.

Theorem 3.11: If $k_{1-2} + k_{2-1} > \min(k_{12-1}, k_{12-2})$ and $k_{1-2} < k_{1-1}$, then the rate pair in the following region can be achieved.

Region 2:

$$D_1 \cap (D_2 \cup D_3 \cup D_4) \quad \text{if } k_{2-1} < k_{2-2}, \text{ or}$$
$$D_1 \cap (D_2 \cup D_3) \quad \text{if } k_{2-1} \geq k_{2-2}, \text{ where}$$

$$\begin{aligned}
&D_1 : R_1 \leq k_{1-1}, \\
&D_2 : R_1 + R_2 \leq rank([H_{11} \quad H_{12}M_2]) \\
&\quad \text{when } R_2 \leq R_2^*, \\
&D_3 : R_1 + 2R_2 \leq R_2^* + rank([H_{11} \quad H_{12}M_2]) \\
&\quad \text{when } R_2^* \leq R_2 \leq \min(k_{2-1}, k_{2-2}), \\
&D_4 : R_1 + R_2 \leq R_2^* + rank([H_{11} \quad H_{12}M_2]) - k_{2-1} \\
&\quad \text{when } k_{2-1} < R_2 \leq k_{2-2},
\end{aligned}$$

where $R_2^* = \min(k_{12-1}, k_{12-2}) - k_{1-2}$, M_1 and M_2 are the encoding matrices at Q_1.

Note that in the above characterization, the rate constraints depend on $rank([H_{11} \quad H_{12}M_2])$; we show a lower bound on $rank([H_{11} \quad H_{12}M_2])$ in Section III-B1.

Proof: Given that $k_{1-2} + k_{2-1} > \min(k_{12-1}, k_{12-2})$ and $k_{1-2} < k_{1-1}$, we will extend the rate region from Q_1 where $R_1^* = k_{1-2}$, $R_2^* = \min(k_{12-1}, k_{12-2}) - k_{1-2}$. Let M_1 and M_2 denote the encoding matrices at Q_1. At Q_1, we first need to increase R_1 while keeping R_2 as large as possible. Suppose that we can use the Rate Increase Lemma to increase R_1. This implies that $\min(k_{12-1}, k_{12-2}) = rank([H_{11}M_1 \quad H_{12}M_2]) < rank([H_{11} \quad H_{12}M_2]) \leq rank([H_{11} \quad H_{12}]) = k_{12-1}$ which implies that $\min(k_{12-2}, k_{12-1}) = k_{12-2}$. In the following discussion, we assume this is the case. By Rate Increase Lemma, we can achieve the rate point $W' = (R_1', R_2') = (rank([H_{11} \quad H_{12}M_2]) - R_2^*, R_2^*)$. The corresponding encoding matrices are M_1' and $M_2' = M_2$.

When we want to further increase R_1 above R_1', we could use Rate Exchange Lemma – 1-1 tradeoff – part (a) repeatedly, since $rank(H_{21}M_1) = k_{1-2} = R_1^*$ and $span(M_1) \subseteq span(M_1')$, implying that $rank(H_{21}M_1') = rank(H_{21}) = k_{1-2}$. When R_1' is increased by δ, R_2' is decreased by δ where $0 \leq \delta \leq \min(R_2^*, k_{1-1} - R_1')$ ($\delta \leq k_{1-1} - R_1'$ comes from the fact that R_1' can be increased to at most k_{1-1}). Terminal t_1 can decode messages from s_1 at rate $R_1'' = R_1' + \delta$ and t_2 can decode messages from s_2 at rate $R_2'' = R_2' - \delta$. Denote the new set of encoding matrices as M_1'' and M_2''. This is shown by the line $(W', \bar{W}')$ in Fig. 4(a) which corresponds to D_2.

On the other hand, at W', we can increase R_2 such that $R_2 = R_2' + \delta_1$ where $0 \leq \delta_1 \leq \min(k_{2-1} - R_2^*, k_{2-2} - R_2^*)$. First note that $k_{12-2} = rank([H_{21}M_1 \quad H_{22}M_2]) \leq rank([H_{21}M_1' \quad H_{22}M_2']) \leq rank([H_{21}M_1' \quad H_{22}]) \leq rank([H_{21} \quad H_{22}]) = k_{12-2}$ which implies $rank([H_{21}M_1' \quad H_{22}M_2']) = rank([H_{21}M_1' \quad H_{22}])$. Then by using Rate Exchange Lemma – 1-2 tradeoff, since $rank(H_{12}) - rank(H_{12}M_2') = k_{2-1} - (\min(k_{12-1}, k_{12-2}) - k_{1-2}) > 0$ we can increase R_2' by δ_1 and decrease R_1' by $2\delta_1$, and the boundary point $(R_1' - 2\delta_1, R_2' + \delta_1)$ can be achieved where $0 \leq \delta_1 \leq \min(k_{2-1} - R_2^*, k_{2-2} - R_2^*, R_1'/2)$ which corresponds to D_3 ($\delta_1 \leq R_1'/2$ comes from the fact that R_1 should be not smaller than 0). If we have that $k_{2-1} \leq \min(k_{2-2}, R_1'/2 + R_2^*)$, we will arrive at the boundary point $W'' = (R_1'', R_2'') = (R_2^* + rank([H_{11} \quad H_{12}M_2]) - 2k_{2-1}, k_{2-1})$. The corresponding matrices are M_1'' and M_2''. This is demonstrated by the line (W', W'') in Fig. 4(a).

If we have that $R_1'' \geq 0$ and $k_{2-1} < k_{2-2}$, at point W'', we can further increase R_2 such that $R_2 = R_2'' + \delta_2$ and $R_1 = R_1'' - \delta_2$ where $0 \leq \delta_2 \leq \min(k_{2-2} - k_{2-1}, R_1'')$. The corresponding encoding matrix at s_2 is M_2'''. By Rate Ex-

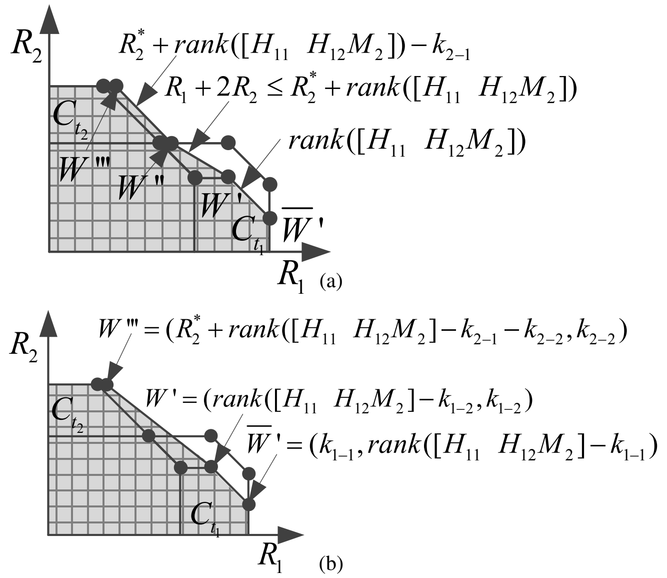

Fig. 4. (a) The extended rate region for the high interference case from point Q_1. (b) The final extended rate region for the case of high interference.

change Lemma – 1-1 tradeoff – part (a), since $rank(H_{12}) = rank(H_{12}M_2'')$, t_1 can decode at rate $R_1'' - \delta_2$, and t_2 can decode at rate $R_2'' + \delta_2$. Then W''' is achieved and the procedure is demonstrated by the line (W'', W''') in Fig. 4(a) which corresponds to D_4. The entire extended rate region for this case is shown in Fig. 4(a). ∎

We next consider increasing R_2 above R_2^{**} at Q_2. If $k_{2-1} \geq k_{2-2}$, R_2 cannot be increased as $R_2^{**} = k_{2-2}$. Hence, we assume that $k_{2-1} < k_{2-2}$. A similar analysis for Q_2 results in the following region.

Corollary 3.12: If $k_{1-2} + k_{2-1} > \min(k_{12-1}, k_{12-2})$ and $k_{2-1} < k_{2-2}$, then the rate pair in the following region can be achieved.

Region 3:

$$\begin{array}{ll} D_1' \cap (D_2' \cup D_3' \cup D_4') & \text{if } k_{1-2} < k_{1-1}, \text{ or} \\ D_1' \cap (D_2' \cup D_3') & \text{if } k_{1-2} \geq k_{1-1} \text{ where,} \end{array}$$

$$\begin{aligned} &D_1' : R_2 \leq k_{2-2}, \\ &D_2' : R_1 + R_2 \leq rank([H_{21}M_1 \;\; H_{22}]) \\ &\quad \text{when } R_1 \leq R_1^{**}, \\ &D_3' : 2R_1 + R_2 \leq R_1^{**} + rank([H_{21}M_1 \;\; H_{22}]) \\ &\quad \text{when } R_1^{**} \leq R_1 \leq \min(k_{1-2}, k_{1-1}), \\ &D_4' : R_1 + R_2 \leq R_1^{**} + rank([H_{21}M_1 \;\; H_{22}]) - k_{1-2} \\ &\quad \text{when } k_{1-2} < R_1 \leq k_{1-1}, \end{aligned}$$

where $R_1^{**} = \min(k_{12-1}, k_{12-2}) - k_{2-1}$, M_1 and M_2 are the encoding matrices at Q_2.

From the above argument, the overall rate region is the convex hull of multicast region, and either Region 2 or Region 3 or both depending upon the cut conditions. For instance when $k_{1-2} < k_{1-1}$ and $k_{2-1} < k_{2-2}$ the final region is shown in Fig. 4(b), where boundary segment $W''' - W'$ is achieved via timesharing.

Finally, note that when $k_{1-2} \geq k_{1-1}$ and $k_{2-1} \geq k_{2-2}$, we cannot enlarge the region using our achievability schemes, i.e., the achievable region is the multicast region.

1) Lower bound of $rank([H_{11} \;\; H_{12}M_2])$: As before, let (R_1^*, R_2^*) denote the rate point at Q_1 and let M_1 and M_2 denote the corresponding encoding matrices. First note that $rank([H_{11} \;\; H_{12}M_2]) \geq rank(H_{11}) = k_{1-1}$ and $rank([H_{11} \; H_{12}M_2]) \geq rank([H_{11}M_1 \; H_{12}M_2]) = R_1^* + R_2^*$. Next we will also find another nontrivial lower bound of $rank([H_{11} \;\; H_{12}M_2])$ by the following lemma.

Lemma 3.13: Given $rank([H_{11} \;\; H_{12}]) = k_{12-1}$, $rank(H_{12}) = k_{2-1}$ and $rank([H_{12}M_2]) = l$, we have $rank([H_{11} \; H_{12}M_2]) \geq k_{12-1} - k_{2-1} + l$.

Proof: By the assumed conditions, there are k_{2-1} columns in H_{12} that are linearly independent, and in H_{11}, we can find a subset of $k_{12-1} - k_{2-1}$ columns denoted H_{11}' such that $span(H_{11}') \cap span(H_{12}) = \{0\}$ and $rank(H_{11}') = k_{12-1} - k_{2-1}$, which further imply that $rank([H_{11}' \; H_{12}]) = k_{12-1}$.

Since $span(H_{12}M_2) \subseteq span(H_{12})$ this means that $span(H_{11}') \cap span(H_{12}M_2) = \{0\}$. Then $rank([H_{11}' \;\; H_{12}M_2]) = rank(H_{11}') + rank(H_{12}M_2) = k_{12-1} - k_{2-1} + l$. Hence, $rank([H_{11} \;\; H_{12}M_2]) \geq rank([H_{11}' \; H_{12}M_2]) = k_{12-1} - k_{2-1} + l$. ∎

Together with the two lower bounds above, we have $rank([H_{11} \; H_{12}M_2]) \geq \max(k_{1-1}, k_{12-1} - k_{2-1} + R_2^*, R_1^* + R_2^*)$. A case where $\max(k_{1-1}, k_{12-1} - k_{2-1} + R_2^*, R_1^* + R_2^*) = k_{12-1} - k_{2-1} + R_2^*$ is shown in Fig. 4(b) where $R_2^* = k_{12-2} - k_{1-2}$.

C. Increasing the achievable rate region by modifying the graph

Thus far, we have presented achievable rate regions for both the low and high interference scenarios. An interesting observation about these regions is that it is possible to enlarge the regions by considering the removal of judiciously chosen edges from the network. We have noted that by removing certain edges from the network, the achievable rate region can be extended. For example, Fig. 5 corresponds to a scenario where $k_{1-1} = 3$, $k_{1-2} = 1$, $k_{2-1} = 3$, $k_{2-2} = 3$, $k_{12-1} = 3$ and $k_{12-2} = 3$. Hence, the sum rate $R_1 + R_2 \leq 3$ using Theorem 3.11. However, one can achieve the rate points $(R_1, R_2) = (1, 3)$ and $(3, 1)$ by removing edges e_1 and e_2 since k_{2-1} drops to 1 and the low interference result (cf. Theorem 3.8) applies. Furthermore note that the rate point $(1, 3)$ is not achievable by routing, i.e., network coding is essential for achieving this point.

In principle, one could consider the union of the achievable rate regions obtained by removing certain subset of the edges from the network to perhaps obtain a larger region. Finding such edges in a systematic manner is an interesting open problem. However, we are unaware of any known algorithm for it.

IV. Achievable Rate Region for Given $k_{1-12}, k_{2-12}, k_{1-1}, k_{2-2}, k_{1-2}$, and k_{2-1}

We have discussed the achievable rate region given $k_{12-1}, k_{12-2}, k_{1-1}, k_{2-2}, k_{1-2}$, and k_{2-1} in the previous section. However, there are other cuts that are potentially

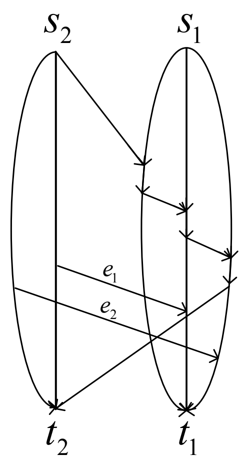

Fig. 5. An example of a network where a larger achievable rate region can be achieved by removing edges e_1 and e_2.

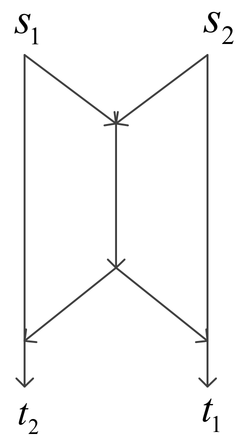

Fig. 6. An example of a network where the achievable rate regions are different using the original result and the reversibility result. All edges are unit capacity.

useful in finding the achievable rate region. In this section, we will discuss the achievable rate region for given $k_{1-12}, k_{2-12}, k_{1-1}, k_{2-2}, k_{1-2}$, and k_{2-1} using the reversibility result introduced in [26]. Towards this end define the reverse of a network G as the network $G' = (V', E')$ where (1) The nodes V' and edges E' in G' are the same as in G, except the direction of edges are reversed. (2) The sources in G are the terminals in G' and vice versa.

For the double unicast problem, we will have that $s'_i = t_i$ and $t'_i = s_i$, $i = 1, 2$. Let $k_{1-12}, k_{2-12}, k_{1-1}, k_{2-2}, k_{1-2}$ and k_{2-1} denote the cut in G and let $k'_{12-1}, k'_{12-2}, k'_{1-1}, k'_{2-2}, k'_{1-2}$ and k'_{2-1} denote the cut in G'. It is evident that $k'_{12-1} = k_{1-12}$, $k'_{12-2} = k_{2-12}$, $k'_{1-1} = k_{1-1}$, $k'_{2-2} = k_{2-2}$, $k'_{1-2} = k_{2-1}$ and $k'_{2-1} = k_{1-2}$. By Theorem 4 in [26] a linear network coding solution for rate pair (R_1, R_2) in the original network G is in one-to-one correspondence with the rate pair $(R'_1, R'_2) = (R_1, R_2)$ in the reversed network G'. Thus, our idea is to determine an achievable rate pair in G' and then claim the existence of a corresponding rate pair in G. The process consists of substituting the corresponding cuts of the reverse network into the multicast region $\mathcal{B}$, Region 1, Region 2 and Region 3 of the original network, to obtain a new set of regions $\mathcal{B}'$, Region 1', Region 2' and Region 3'.

In the interest of avoiding repetitive arguments, we discuss the process of determining Region 2' by means of an example. For the original graph, in Region 2, $D_2 : R_1 + R_2 \leq rank([H_{11} \quad H_{12}M_2])$ when $R_2 \leq \min(k_{12-1}, k_{12-2}) - k_{1-2}$. Thus, for Region 2', the corresponding $D_2 : R_1 + R_2 \leq rank([H'_{11} \quad H'_{12}M'_2])$ when $R_2 \leq \min(k_{1-12}, k_{2-12}) - k_{2-1}$ where H'_{ij} is the transfer matrix from s'_j to t'_i, and M'_i is the source encoding matrix at s'_i. The other inequalities can be determined in a similar manner.

Hence, given all possible cuts in a double unicast network, the achievable rate region is convex hull of multicast region $\mathcal{B}$, $\mathcal{B}'$ and the corresponding extended region in different cases.

In order to demonstrate the utility of considering the reversed network, consider the network shown in Fig. 6. It can be verified that the rate regions are different using the original result and reversibility result. with our schemes. In particular, using the reversibility result can achieve rate point (1,1) whereas the original result cannot.

V. Comparison with Existing Results

The work that is most closely related to the present paper is by [14] that also considers the double unicast problem with arbitrary rates. Assuming that $k_{2-2} \leq k_{1-1}$, the region in [14] is given by EF09 = EF09(a) ∪ EF09(b), where

$$\text{EF09(a)} = \{(R_1, R_2) : R_1 + 2R_2 \leq k_{1-1}, R_2 \leq k_{2-2}\}, \text{and}$$
$$\text{EF09(b)} = \{(R_1, R_2) : 2R_1 + R_2 \leq k_{2-2}, R_1 \leq k_{1-1}\}.$$

A comparison between our region and theirs indicates that our region is larger than theirs. To see this, consider the low interference case and a rate point (R_1, R_2) that lies in EF09(a). We have that $R_1 + R_2 \leq R_1 + 2R_2 \leq k_{1-1} \leq k_{12-1} - k_{2-1} + k_{12-2} - k_{1-2}$ (since $k_{1-2} + k_{2-1} \leq \min(k_{12-1}, k_{12-2})$) and $R_2 \leq k_{2-2}$, i.e. (R_1, R_2) also belongs to our region.

For the high interference case, we argue as follows. Let (R_1, R_2) belong to EF09(a).

- If $k_{1-2} \leq k_{1-1}$, we show that (R_1, R_2) belongs to Region 2. Note that $R_1 + 2R_2 \leq k_{1-1} \leq rank([H_{11} \quad H_{12}M_2])$. However, the RHS of D_2 and D_3 is at least as large as $rank([H_{11} \quad H_{12}M_2])$, and for D_4 we have $R_1 + 2R_2 \leq rank([H_{11} \quad H_{12}M_2]) \leq R_2^* + rank([H_{11} \quad H_{12}M_2]) - k_{2-1} + R_2$ (since in D_4, $k_{2-1} < R_2 \leq k_{2-2}$) indicating that (R_1, R_2) is within Region 2.
- If $k_{1-2} > k_{1-1}$ and $k_{2-1} \geq k_{2-2}$, we have $R_1 + R_2 \leq R_1 + 2R_2 \leq k_{1-1} \leq \min(k_{1-2}, k_{12-1}) \leq \min(k_{12-2}, k_{12-1})$ which shows that (R_1, R_2) is within our multicast region.
- If $k_{1-2} > k_{1-1}$ and $k_{2-1} < k_{2-2}$, we consider different ranges for R_2. For $0 \leq R_2 \leq k_{2-1}$, $R_1 + R_2 \leq R_1 + 2R_2 \leq k_{1-1} \leq \min(k_{1-2}, k_{12-1}) \leq \min(k_{12-2}, k_{12-1})$ which implies that (R_1, R_2) is within our multicast region. On the other hand when $k_{2-1} \leq R_2 \leq k_{2-2}$, we have $k_{1-1} - 2k_{2-2} \leq R_1 \leq$

$k_{1-1} - 2k_{2-1}$ (from the definition of EF09(a)). This implies that (R_1, R_2) belongs to Region 3. To see this we note that the relevant range of Region 3 is D_2' since $k_{1-1} - 2k_{2-1} \leq \min(k_{12-1}, k_{12-2}) - k_{2-1}$. We have $R_1 + R_2 \leq R_1 + 2R_2 \leq k_{1-1} \leq \min(k_{1-1} + k_{2-1}, \min(k_{12-1}, k_{12-2})) = R_1^{**} + R_2^{**} = rank([H_{21}M_1 \quad H_{22}M_2]) \leq rank([H_{21}M_1 \quad H_{22}])$ indicating that such a point is within Region 3.

In a similar manner it can be shown that all rate points in EF09(b) are within our rate region.

The authors in [12] and [13] explore the unit-rate case $R_1 = R_2 = 1$ in detail. Such schemes can potentially be packed into networks with higher capacities. References [12], [13] rely heavily on an analysis of the graph theoretic structures that are possible in double unicast networks. Thus, our scheme will in general be weaker than their approach on certain networks. Likewise the work of [8] [9] also considers the achievable rate region using network coding between pair of sources. However, there are networks where our approach is strictly better than all the above approaches. We show such an example in Fig. 7. In Fig. 7, we can achieve rates (4,2) by the argument using in Region 2, whereas it can be verified that the above schemes do not support this rate point. For instance, if $R_2 = 2$, $R_1 \leq 3$ in EF09, whereas the scheme in [12] can at most achieve a rate of $(1, 2)$. Furthermore, we note that the enlargement of the achievable region by considering the removal of certain edges discussed in Section III-C also improves our region in many cases.

The following results have appeared since the submission of the present paper and the publication of our preliminary conference paper [21]. The work of [22] treats the two unicast problem as an instance of a linear deterministic interference channel and finds a network code that uses random linear network coding. Their region contains our proposed achievable region. The authors in [23] also derive an achievable region by exploiting the equivalence with deterministic interference channels; their region is completely specified by the cut values in the network (in contrast, in certain cases our region and the region in [22] is specified in terms of the rank of matrices that depend on the network code). However, for some networks our scheme achieves a larger region. As an example, if one considers the two-unicast butterfly network with $k_{1-1} = k_{2-2} = 1, k_{1-2} = k_{2-1} = 2$ and $k_{12-1} = k_{12-2} = 2$, our scheme achieves the multicast point $(1, 1)$ whereas the region in [23] is empty.

VI. Conclusions and Future Work

In this work, we presented an achievable rate region for the double unicast problem for directed acyclic networks with unit capacity edges. The proposed strategy combines random linear network coding along with appropriate precoding at the source nodes. Networks are classified according the relationship of the values of the cuts between various subsets of the sources and the terminals. We begin with the multicast region where both sources are multicast to both terminals and then enlarge the region by either unilaterally increasing one of the rates or trading off rates between the connections. The proposed region can potentially be enlarged by considering regions that are obtained by the judicious removal of certain edges from the network. Future work would include the investigation of systematic techniques for finding the appropriate edges to be removed.

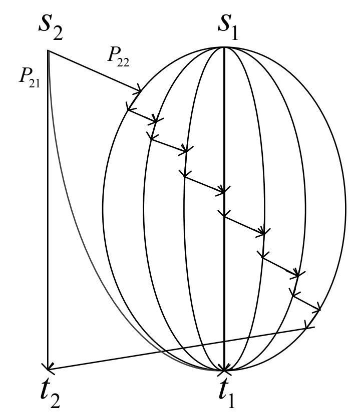

Fig. 7. An example of a high interference network when our scheme can achieve a higher rate pair compared to many other schemes.

Appendix

Lemma A.1: If $rank(HM) = rank(H) = r$, then $span(HM) = span(H)$.

Proof: First note that $span(HM) \subseteq span(H)$. Assume $span(HM) \neq span(H)$, then there is a vector $\vec{v} \in span(H)$ but not in $span(HM)$. Then,

$$rank([HM \quad \vec{v}]) = rank(HM) + 1 = r + 1 > r = rank(H)$$

However, it contradicts the fact that $rank(H) \geq rank([HM \quad \vec{v}])$, since $[HM \quad \vec{v}] \subseteq span(H)$. Hence $span(HM) = span(H)$. ■

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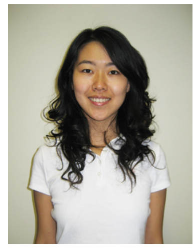

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