Rate and power allocation under the pairwise distributed source coding constraint

Shizheng Li and Aditya Ramamoorthy
Department of Electrical and Computer Engineering
Iowa State University
Ames, Iowa 50011
Email: {szli, adityar}@iastate.edu

***Abstract*—We explore the problem of rate and power allocation for a sensor network where pairwise distributed source coding is employed (introduced by Roumy and Gesbert '07). For noiseless node-terminal channels, we show that the minimum sum rate assignment with this property can be found by finding a minimum weight arborescence in an appropriately defined directed graph. For orthogonal noisy node-terminal channels, the minimum sum power allocation can be found by finding a minimum weight matching forest in a mixed graph. Numerical results are presented for the noiseless case showing that our solution outperforms previous solutions when source correlations are high.**

I. Introduction

The availability of low-cost sensors has enabled the emergence of large-scale sensor networks in recent years. Sensor networks typically consist of sensors that have limited power and are moreover energy constrained since they are usually battery-operated. The data that is sensed by sensor networks and communicated to a terminal is usually correlated. Thus, for sensor networks it is important to allocate resources such as rates and power by taking the correlation into account.The famous Slepian-Wolf theorem [1] shows that the distributed compression (or distributed source coding) of correlated sources can in fact be as efficient as joint compression. Coding techniques that approach the Slepian-Wolf bounds have been investigated [2] and their usage proposed in sensor networks [3]. Typically one wants to minimize metrics such as the total rate or total power expended by the sensors in such situations. A number of authors have considered problems of this flavor [4], [5]. These papers assume the existence of Slepian-Wolf codes that work for a large number of sensors.

In practice, the design of low-complexity Slepian-Wolf codes (e.g. [6]) is well understood only for the case of two sources X and Y and there have been constructions that are able to operate on the boundary of the Slepian-Wolf region. In particular, the design of codes is easiest for the corner points where the rate pair is either $(H(X), H(Y|X))$ or $(H(X|Y), H(Y))$. Although there are some coding schemes exist for multiple sources, for example [7], these require strict assumptions on the correlation model. Thus, given the current state of the art in code design it is of interest to consider coding strategies for sensor networks where pairs of nodes can be decoded at a time instead of all at once. This observation was made in the work of Roumy and Gesbert in [8]. In that work they formulated the pairwise distributed source coding problem and presented algorithms for rate and power allocation under different scenarios. In particular, they considered the case when there exist direct channels between each source node and the terminal. Furthermore, the terminal can only decode the sources pairwise. We briefly review their work below. The work of [8] considers two cases.

i) *Case 1 - Noiseless node-terminal channels.* Under this scenario, they considered the problem of deciding which particular nodes should be decoded together at the terminal and their corresponding rate allocations so that the total sum rate is minimized.
ii) *Case 2 - Orthogonal noisy node-terminal channels.* In this case the channels were assumed to be noisy and orthogonal and the objective was to decide which nodes would be paired and their corresponding power allocation.

In [8], the problem was mapped onto the problem of choosing the minimum weight matching [9] of an appropriately defined weighted undirected graph.

In this paper we consider a class of pairwise distributed source coding solutions that is larger than the ones considered in [8]. A simple example demonstrates that it is not necessary to only consider matchings if one is interested in exploiting the fact that practical Slepian-Wolf solutions are for pairs of sources. Consider four correlated sources X_1, X_2, X_3 and X_4. The solution of [8] constructs a complete graph on the four nodes $X_1, \ldots, X_4$ and assigns the edge weights as the joint entropies i.e. the edge (X_i, X_j) is assigned weight $H(X_i, X_j)$. A minimum weight matching algorithm is then run on this graph to find the minimum sum rate and rate allocation. Suppose that this yields the matching (X_1, X_3) and (X_2, X_4) so that the sum rate becomes

$$\sum_{i=1}^{4} R_i = H(X_1, X_3) + H(X_2, X_4).$$

Since conditioning reduces entropy, it is simple to observe that

$$\begin{aligned} &H(X_1, X_3) + H(X_2, X_4) \\ &\geq H(X_1) + H(X_3|X_1) + H(X_2|X_3) + H(X_4|X_2). \end{aligned}$$

We now show that an alternative rate allocation: $R_1 = H(X_1), R_2 = H(X_2|X_3), R_3 = H(X_3|X_1)$ and $R_4 =$

This research was supported in part by NSF grant CNS-0721453.

$H(X_4|X_2)$ can still allow pairwise decoding of the sources at the terminal. Note that at the decoder we have,

a) X_1 is known since $R_1 = H(X_1)$.
b) X_3 can be recovered since X_1 is known and the decoder has access to $H(X_3|X_1)$ amount of data.
c) X_2 can be recovered since X_3 is known (from above) and the decoder has access to $H(X_2|X_3)$ amount of data.
d) Similarly, X_4 can be recovered.

As we see above, the sources can be decoded at the terminal in a pipelined manner. The method of source-splitting [10], [11] is closely related to this approach. Given M sources and an arbitrary rate point in their Slepian-Wolf region, it converts the problem into a rate allocation at a Slepian-Wolf corner point for appropriately defined $2M-1$ sources. However as pointed out before, code designs even for corner points are not that well understood for more than two sources. Thus, while using source-splitting can result in sum-rate optimality i.e. the sum rate is the joint entropy, it may not be very practical given the current state of the art. Moreover, for M sources it requires the design of approximately twice as many encoders and more decoding sub-modules that also comes at the cost of complexity.

In this paper, motivated by complexity issues, we present an alternate formulation of the pairwise distributed source coding problem that is more general than [8]. We demonstrate that for noiseless channels the minimum sum rate allocation problem becomes one of finding a minimum weight arborescence of an appropriately defined directed graph. Next, we show that in the case of noisy channels, the minimum sum power allocation problem can be mapped onto finding the minimum weight matching forest of an appropriately defined mixed graph[1]. Simulation results show that our solutions are significantly better than those in [8] in the cases when correlations are high.

This paper is organized as follows. In Section II and III we present our solution for noiseless channels and noisy channels respectively. Numerical results for the noiseless case are given in Section IV and Section V concludes this paper.

II. Noiseless case

Consider a set of correlated sources $X_1, X_2, \ldots, X_n$ in a sensor network. The source X_i encodes its data at a rate denoted R_i (in bits per symbol). We assume that there is a direct noiseless channel between each source and the terminal. The rate allocation vector is denoted by $\mathbf{R} = (R_1, R_2, \ldots, R_n)$. We are interested in finding a rate allocation that minimizes the sum rate $\sum_{i=1}^{n} R_i$ subject to the constraint that the sources can be reconstructed at the terminal. Moreover we require that the decoder be able to decode two sources at a time. We denote the Slepian-Wolf region of the pair (X_i, X_j) by SW_{ij} which is given by

$$SW_{ij} \triangleq \left\{ \begin{array}{cr} & R_i \geq H(X_i|X_j) \\ (R_i, R_j) & R_j \geq H(X_j|X_i) \\ & R_i + R_j \geq H(X_i, X_j) \end{array} \right\}. \quad (1)$$

We now formulate the pairwise decoding constraint mathematically. Let $[n]$ denote the index set $\{1, \ldots, n\}$.

[1]A mixed graph has both directed and undirected edges

Definition 1: Pairwise property of rate assignment. Consider a set of discrete memoryless sources $X_1, X_2, \ldots, X_n$ and the corresponding rate assignment $\mathbf{R} = (R_1, R_2, \ldots, R_n)$. The rate assignment is said to satisfy the pairwise property if for each source $X_i, i \in [n]$, there exists an ordered sequence of sources $(X_{i_1}, X_{i_2}, \ldots, X_{i_k})$ such that

$$R_{i_1} \geq H(X_{i_1}), \quad (2)$$

$$R_{i_j} \geq H(X_{i_j}|X_{i_{j-1}}), \quad \text{for } 2 \leq j \leq k, \text{ and} \quad (3)$$

$$R_i \geq H(X_i|X_{i_k}) \quad (4)$$

Note that a rate assignment that satisfies the pairwise property allows the possibility that each source can be reconstructed at the decoder by solving a sequence of decoding operations at the SW corner points e.g. for decoding source X_i one can use X_{i_1} (since $R_{i_1} \geq H(X_{i_1})$), then decode X_{i_2} using the knowledge of X_{i_1}. Continuing in this manner finally X_i can be decoded. A rate assignment $\mathbf{R}$ shall be called pairwise valid (or valid in this section), if it satisfies the pairwise property.

An equivalent definition can be given in graph-theoretic terms by constructing a graph called the pairwise property test graph corresponding to the rate assignment.

1) Inputs : the number of nodes n, $H(X_i)$ for all $i \in [n]$, $H(X_i|X_j)$ for all $i, j \in [n]^2$ and the rate assignment $\mathbf{R}$.
2) Initialize a graph $G = (V, A)$ with a total of $2n$ nodes i.e. $|V| = 2n$. There are n *regular* nodes denoted $1, 2, \ldots, n$ and n *starred* nodes denoted $1^*, 2^*, \ldots, n^*$.
3) For each $i \in [n]$ perform the following steps. Let $W_A(j \to i)$ denote the weight on directed edge $(j \to i)$.
   i) If $R_i \geq H(X_i)$ then insert edge $(i^* \to i)$ with $W_A(i^* \to i) = H(X_i)$.
   ii) If $R_i \geq H(X_i|X_j)$ then insert edge $(j \to i)$ with $W_A(j \to i) = H(X_i|X_j)$.
4) Remove all nodes that do not participate in any edge i.e. they have neither incoming nor outgoing edges.

We denote the resulting graph for a given rate allocation by $G(\mathbf{R}) = (V, A)$. Note that if $\mathbf{R}$ is valid, the graph still contains at least one starred node. Next, based on $G(\mathbf{R})$ we define a set of nodes that are called the parent nodes.

$$\text{Parent}(\mathbf{R}) = \{i^* | (i^* \to i) \in A\}$$

i.e. Parent($\mathbf{R}$) corresponds to the starred nodes for the set of sources for which the rate allocation is at least the entropy. Mathematically if $i^* \in \text{Parent}(\mathbf{R})$, then $R_i \geq H(X_i)$. We now demonstrate the equivalence between the pairwise property and the construction of the graph above. The proof is omitted due to lack of space.

Lemma 1: Consider a set of discrete correlated sources $X_1, \ldots X_n$ and a corresponding rate assignment $\mathbf{R} = (R_1, \ldots, R_n)$. Suppose that we construct $G(\mathbf{R})$ based on the algorithm above. The rate assignment $\mathbf{R}$ satisfies the pairwise property if and only if for all regular nodes $i \in V$ there exists a starred node $j^* \in \text{Parent}(\mathbf{R})$ such that there exists directed path from j^* to i in $G(\mathbf{R})$.

We now proceed to define another set of graphs that shall be useful for presenting the main result of this section.

Definition 2: Specification of $G_{i^*}(\mathbf{R})$. Suppose that we construct graph $G(\mathbf{R})$ as above and find Parent($\mathbf{R}$). For each

$i^* \in \text{Parent}(\mathbf{R})$ we construct $G_{i^*}(\mathbf{R})$ in the following manner: For each $j^* \in \text{Parent}(\mathbf{R})\backslash\{i^*\}$ remove the edge $(j^* \to j)$ and the node j^* from $G(\mathbf{R})$.

For the next result we need to introduce the concept of the arborescence or directed spanning tree of a graph (see [9]).

Definition 3: An *arborescence (also called directed spanning tree)* of a directed graph $G = (V, A)$ rooted at vertex $r \in V$ is a subgraph T of G such that it is a spanning tree if the orientation of the edges is ignored and there is a path from r to all $v \in V$ when the direction of edges is taken into account.

Theorem 1: Consider a set of discrete correlated sources $X_1, \dots, X_n$ and let the corresponding rate assignment $\mathbf{R}$ be pairwise valid. Let $G(\mathbf{R})$ be constructed as above. There exists another valid rate assignment $\mathbf{R}^{'}$ that can be described by the edge weights of an arborescence of $G_{i^*}(\mathbf{R})$ rooted at i^* where $i^* \in \text{Parent}(\mathbf{R})$ such that $R_j^{'} \leq R_j$, for all $j \in [n]$.

Proof: We shall show that a new graph can be constructed from which $\mathbf{R}^{'}$ can be obtained. This shall be done by a series of graph-theoretic transformations.

1) Pick an arbitrary starred node $j^* \in \text{Parent}(\mathbf{R})$ and construct $G_{j^*}(\mathbf{R})$. We claim that in the current graph $G_{j^*}(\mathbf{R})$ there exists a path from the starred node j^* to all regular nodes $i \in [n]$. To see this note that since $\mathbf{R}$ is pairwise valid, for each regular node i there exists a path from some starred node to i in $G(\mathbf{R})$. If for some regular node i, the starred node is j^*, the path is still in $G_{j^*}(\mathbf{R})$. Now consider a regular node i_1 and suppose there exists a directed path $k^* \to k \to \beta_1 \ldots \to i_1$ in $G(\mathbf{R})$ where $k^* \in \text{Parent}(\mathbf{R}), k^* \neq j^*$. Since $k^* \in \text{Parent}(\mathbf{R})$,
$$R_k \geq H(X_k) \geq H(X_k|X_l) \quad \forall l \in [n] \tag{5}$$
This implies that $(l \to k) \in G_{j^*}(\mathbf{R}), \forall l \in [n]$, in particular, $(j \to k) \in G_{j^*}(\mathbf{R})$. Therefore, in $G_{j^*}(\mathbf{R})$ there exists the path $j^* \to j \to k \to \beta_1 \ldots \to i_1$.
This claim implies that there exists an arborescence rooted at j^* in $G_{j^*}(\mathbf{R})$ [9].
2) Suppose we find such one such arborescence T_{j^*} of $G_{j^*}(\mathbf{R})$. In T_{j^*} every node except j^* has exactly one incoming edge (by the property of an arborescence [9]). Let $inc(i)$ denote the node such that $(inc(i) \to i) \in T_{j^*}$. We now define a new rate assignment $\mathbf{R}^{'}$ given by
$$R_i^{'} = W_A(inc(i) \to i) \tag{6}$$
$$= H(X_i|X_{inc(i)}) \quad \text{for } i \in [n] \text{ and } i \neq j \tag{7}$$
$$R_j^{'} = W_A(j^* \to j) = H(X_j). \tag{8}$$
The existence of edge $(j^* \to j) \in G(\mathbf{R})$ implies $R_j^{'} = H(X_j) \leq R_j$. Similarly, we have $R_i^{'} \leq R_i$ for $i \in [n]\backslash\{j\}$. And it is easy to see that $\mathbf{R}^{'}$ is a valid rate assignment. ■

Thus, the above theorem implies that valid rate assignments that are described on arborescences of the graphs $G_{i^*}(\mathbf{R})$ are the best from the point of view of minimizing the sum rate. Finally we have the following theorem that says that the valid rate assignment that minimizes the sum rate can be found by finding minimum cost arborescences of appropriately defined graphs. For the statement of the theorem we need to define the following graphs.

a) The graph $G^{tot} = (V^{tot}, A^{tot})$ is such that V consists n regular nodes denoted $1, 2, \dots, n$ and n starred nodes denoted $1^*, 2^*, \dots, n^*$ so that $|V^{tot}| = 2n$. The edge set A^{tot} consists of edges $(i^* \to i), W_A(i^* \to i) = H(X_i)$ for $i \in [n]$ and edges $(i \to j), W_A(i \to j) = H(X_j|X_i)$ for all $i, j \in [n]^2$.
b) For each $i = 1, \dots, n$ we define G_{i^*} as the graph obtained from G^{tot} by deleting all edges of the form $(j^* \to j)$ for $j \neq i$ and all nodes in $\{1^*, \dots, n^*\}\backslash\{i^*\}$.

Theorem 2: Consider a set of sources $X_1, \dots, X_n$. Suppose that we are interested in finding a valid rate assignment $\mathbf{R} = (R_1, \dots, R_n)$ for these sources so that the sum rate $\sum_{i=1}^n R_i$ is minimum. Let $\mathbf{R}^{i^*}$ denote the rate assignment specified by the minimum cost arborescence of G_{i^*}. Then the optimal valid rate assignment can be found as
$$R_{opt} = \arg \min_{i \in \{1, \dots, n\}} \sum_{j=1}^n R_j^{i^*} \tag{9}$$

Proof. From Theorem 1 we have that any valid rate assignment $\mathbf{R}$ can be transformed into new rate assignment that can be described on an arborescence of $G_{i^*}(\mathbf{R})$ rooted at i^* which is component-wise lower than $\mathbf{R}$. This implies that if we are interested in a minimum sum rate solution, it suffices to focus our attention on solutions specified by all solutions that can be described by all possible arborescences of graphs of the form $G_{i^*}(\mathbf{R})$ over all $i^* = 1^*, \dots, n^*$ and all possible valid rate assignments $\mathbf{R}$.

Now consider the graph G_{i^*} defined above. We note that all graphs of the form $G_{i^*}(\mathbf{R})$ where $\mathbf{R}$ is valid are subgraphs of G_{i^*}. Therefore finding the minimum cost arborescence of G_{i^*} will yield us the best rate assignment possible within the class of solutions specified by $G_{i^*}(\mathbf{R})$. Next, we find the best solutions $\mathbf{R}^{i^*}$ for all $i \in [n]$ and pick the solution with the minimum cost. This yields the optimal rate assignment. ■

III. Noisy case

In this section we consider the case when the sources are connected to the terminal by orthogonal noisy channels. The capacity of the channel between node i and the terminal with transmission power P_i and channel gain γ_i is
$$C_i(P_i) \triangleq \log(1 + \gamma_i P_i), \tag{10}$$
where the noise power is normalized to one and channel gains are constants known to the terminal. Therefore, rate R_i should satisfy $R_i \leq C_i(P_i)$. The transmission power is constrained by a maximum power constraint, $P_i \leq P_{max}, i \in [n]$. The objective is to find a rate and power assignment that minimizes the sum power , i.e., $\min \sum_{i=1}^n P_i$ while ensuring that the sources can be recovered at the terminal and that the decoder only decodes two sources at a time. It is easy to see that at the optimum $R_i^* = C_i(P_i^*)$ i.e. the inequality constraint is met with equality. Thus, the power assignment is given by the inverse function of C_i which we denote by $Q_i(R_i^*)$ i.e. $P_i^* = Q_i(R_i^*) = (2^{R_i^*} - 1)/\gamma_i$. The feasible rate region for the node pair (i, j) is the intersection of SW_{ij} and capacity region $C_{ij}(P_i, P_j) \triangleq \{(R_i, R_j) : R_i \leq C_i(P_i), R_j \leq C_j(P_j)\}$.

The solution presented in [8] goes as follows.

1) Find the rate-power allocations over all possible node pairs: $\forall (i, j) \in [n]^2$ such that $i < j$

$$(R^*_{ij}(i), R^*_{ij}(j)) = \arg\min Q_i(R_{ij}(i)) + Q_j(R_{ij}(j))$$
$$\text{s.t. } (R_{ij}(i), R_{ij}(j)) \in SW_{ij} \cap C_{ij}(P_{max}, P_{max}) \quad (11)$$

The power allocations are given by $P^*_{ij}(i) = Q_i(R^*_{ij}(i))$ and $P^*_{ij}(j) = Q_j(R^*_{ij}(j))$[2].

2) Construct an undirected complete graph $G = (V, E)$, where $W_E(i, j) = P^*_{ij}(i) + P^*_{ij}(j)$, and find the minimum weight matching $\mathcal{P}$ in G. The final power allocation for node pair $(i, j) \in \mathcal{P}$ denoted by (P_i, P_j) is $(P^*_{ij}(i), P^*_{ij}(j))$.

The solution for the first step (11) is given in [8] and denoted as $(P^*_{ij}(i), P^*_{ij}(j), R^*_{ij}(i), R^*_{ij}(j))$. In this case, the rate assignments for i and j don't necessarily happen at the corner of the SW bound, i.e., $R^*_{ij}(i)$ may not equal to $H(X_i)$ and the problem is more complicated than noiseless case.

We now present our solution for this case. For a given rate assignment $\mathbf{R}$, we say that X_i is *initially decodable* if $R_i \geq H(X_i)$, or together with another source X_j, $(R_i, R_j) \in SW_{ij}$. Obviously, an initially decodable source can be recovered at the sink. In addition, if we use previously decoded source data as we did in noiseless case, starting with an initially decodable source, more sources can potentially be recovered. We now introduce the generalized pairwise property.

Definition 4: Generalized pairwise property of rate assignment. Consider a set of discrete memoryless sources $X_1, X_2, \dots, X_n$ and the corresponding rate assignment $\mathbf{R} = (R_1, R_2, \dots, R_n)$. The rate assignment is said to satisfy the generalized pairwise property if for each source $X_i, i \in [n]$, X_i is initially decodable or, there exists an ordered sequence of sources $(X_{i_1}, X_{i_2}, \dots, X_{i_k})$ such that

$$X_{i_1} \text{ is initially decodable,} \quad (12)$$
$$R_{i_j} \geq H(X_{i_j}|X_{i_{j-1}}) \quad \text{for } 2 \leq j \leq k. \quad (13)$$
$$R_i \geq H(X_i|X_{i_k}) \quad (14)$$

A rate assignment $\mathbf{R}$ shall be called generalized pairwise valid (or valid in this section), if it satisfies the generalized pairwise property and for every rate $R_i \in \mathbf{R}$, $Q_i(R_i) \leq P_{max}$. A valid rate assignment allows every source to be recovered at the sink. A power assignment $\mathbf{P} = (P_1, P_2, \dots, P_n)$ shall be called valid, if the corresponding rate assignment is valid.

We can rephrase this definition using a graph called generalized pairwise property test graph constructed below.

The input and initialization are the same as pairwise property test graph construction. For each $i \in [n]$ perform the following steps.

i) If $R_i \geq H(X_i)$ then insert directed edge $(i^* \to i)$ with weight $W_A(i^* \to i) = Q_i(H(X_i))$.
ii) If $R_i \geq H(X_i|X_j)$ then insert directed edge $(j \to i)$ with weight $W_A(j \to i) = Q_i(H(X_i|X_j))$.
iii) If $(R_i, R_j) \in SW_{ij}$, then insert undirected edge (i, j) with weight $W_E(i, j) = Q_i(R^*_{ij}(i)) + Q_j(R^*_{ij}(j)) = P^*_{ij}(i) + P^*_{ij}(j)$.

Finally, remove all nodes that do not participate in any edge. We denote the resulting graph for a given rate allocation by $G_T(\mathbf{R}) = (V, E, A)$, where E is undirected edge set and A is directed edge set. Denote the regular node set as $V_R \subset V$.

[2]Here we use subscript ij because $R^*_{ij}(i)$ denotes the rate allocated to node i if i and j are paired. The decision on pairing of nodes is taken in the next step.

Lemma 2: Consider a set of discrete correlated sources $X_1, \dots X_n$ and a corresponding rate assignment $\mathbf{R} = (R_1, \dots, R_n)$. Suppose that we construct $G_T(\mathbf{R})$ based on the algorithm above. The rate assignment $\mathbf{R}$ is generalized pairwise valid if and only if, $\forall R_i \in \mathbf{R}, Q_i(R_i) \leq P_{max}$, and for all regular nodes $i \in V_R$, at least one of these conditions holds:

1) i participates in an undirected edge (i, i');
2) There exists a starred node j^* such that there is a directed path from j^* to i;
3) There exists a regular node j participating in edge (j, j') such that there is a directed path from j to i;

Now, we introduce some definition crucial to the rest of the development.

Definition 5: Given a mixed graph $G = (V, E, A)$, if $e = (i \to j) \in A$, i is the tail and j is the head of e. If $e = (i, j) \in E$, we call both i and j the head of e. For a node $i \in V$, $h_G(i)$ denotes the number of edges for which i is the head.

Definition 6: The *underlying undirected graph* of a mixed graph G denoted by $UUG(G)$ is the undirected graph obtained from the mixed graph by forgetting the orientations of the directed edges, i.e., treating directed edges as undirected edges.

Definition 7: Given a mixed graph $G = (V, E, A)$, a subset $F \in E \cup A$ is called a *matching forest* [12] if F contains no cycles in $UUG(F)$ and any node $i \in V$ is the head of at most one edge in F, i.e.,$\forall i \in V, h_F(i) \leq 1$.

In the context of this section we also define a strict matching forest. For a mixed graph G containing regular nodes and starred nodes, a matching forest F satisfying $h_F(i) = 1, \forall i \in V_R$ (i.e. every regular node is the head of exactly one edge) is called a *strict matching forest(SMF)*.

In the noisy case, SMF plays a role similar to the arborescence in the noiseless case. Now, we introduce a theorem similar to theorem 1.

Theorem 3: Given a generalized pairwise valid rate assignment $\mathbf{R}$ and corresponding power assignment $\mathbf{P}$, let $G_T(\mathbf{R})$ be constructed as above. There exists another valid rate assignment $\mathbf{R}'$ and power assignment $\mathbf{P}'$ that can be described by the edge weights of a strict matching forest of $G_T(\mathbf{R})$ such that $\sum_{i=1}^n P'_i \leq \sum_{i=1}^n P_i$.

Proof. In order to find such a SMF, we first change the weights of $G_T(\mathbf{R})$, yielding a new graph $G'_T(\mathbf{R})$. Let $W'_A(i \to j), W'_E(i, j)$ denote weights in $G'_T(\mathbf{R})$. A weight transformation is done on all edges:

$$W'_E(i, j) = 2\Lambda - W_E(i, j) \quad (15)$$
$$W'_A(i \to j) = \Lambda - W_A(i \to j) \quad (16)$$

where Λ is a sufficient large constant. Next, we find a maximum weight matching forest of $G'_T(\mathbf{R})$. This can be done in polynomial time [12]. Now we have a lemma whose proof is skipped due to space reasons.

Lemma 3: The maximum weight matching forest F_M in $G'_T(\mathbf{R})$ is a strict matching forest, i.e., it satisfies: $\forall i \in V_R, h_{F_M}(i) = 1$.

Note that each regular node is head of exact one edge in F_M. The power allocation is performed as follows. Any $i \in V_R$ is the head of one of three kinds of edges in F_M corresponding to three kinds of rate-power assignment:

1) If there exists a directed edge $(i^* \to i)$, then set $P_i^{'} = Q_i(H(X_i))$ and $R_i^{'} = H(X_i)$. The existence of edge $(i^* \to i)$ in $G_T(\mathbf{R})$ means that $R_i \geq H(X_i)$, so $R_i^{'} \leq R_i$ and $P_i^{'} \leq P_i \leq P_{max}$.
2) If there exists an undirected edge (i, j), set $P_i^{'} = P_{ij}^*(i)$ and $R_i^{'} = R_{ij}^*(i)$. The existence of edge (i, j) in $G_T(\mathbf{R})$ means that R_i and R_j are in the SW region, $P_i \leq P_{max}$ and $P_j \leq P_{max}$. Certainly, in this case, $P_j^{'} = P_{ij}^*(j)$ and $R_j^{'} = R_{ij}^*(j)$, since j is not head of any other edges. We know that $P_{ij}^*(i), P_{ij}^*(j)$ is the minimum sum power solution for node i and j when the rate allocation is in SW region and the power allocation satisfies P_{max} constraints. So $P_i^{'} + P_j^{'} \leq P_i + P_j$, $P_i^{'} \leq P_{max}$, $P_j^{'} \leq P_{max}$.
3) If there exists a directed edge $(j \to i)$, set $P_i^{'} = Q_i(H(X_i|X_j))$ and $R_i^{'} = H(X_i|X_j)$. The existence of edge $(j \to i)$ in $G_T(\mathbf{R})$ means that $R_i \geq H(X_i|X_j)$, so $R_i^{'} \leq R_i$ and $P_i^{'} \leq P_i \leq P_{max}$.

Therefore, the new power allocation $\mathbf{P}^{'}$ reduces the sum power. Notice that when we are assigning new rates to the nodes, the conditions in definition 4 still hold. So the new rate $\mathbf{R}^{'}$ is also valid. So $\mathbf{P}^{'}$ is a valid power allocation with less sum power. ■

The following theorem says that the valid power assignment that minimizes the sum power can be found by finding minimum weight SMF of an appropriately defined graph.

The graph $G^{tot} = (V^{tot}, A^{tot}, E^{tot})$ is such that V^{tot} consists n regular nodes denoted by $1, \ldots, n$ and n starred nodes denoted by $1^*, \ldots, n^*$ so that $|V^{tot}| = 2n$. The directed edge set A^{tot} consists of edges $(i^* \to i), W_A(i^* \to i) = Q_i(H(X_i))$ for $\{i : i \in [n]$ and $Q_i(H(X_i)) \leq P_{max}\}$, and directed edges $(i \to j), W_A(i \to j) = Q_j(H(X_j|X_i))$ for $\{i, j : i, j \in [n]^2$ and $Q_j(H(X_j|X_i)) \leq P_{max}\}$. The undirected edge set E^{tot} consists of edges $(i, j), W_E(i, j) = P_{ij}^*(i) + P_{ij}^*(j)$ for all $i, j \in [n]^2$.

Theorem 4: Consider a set of sources $X_1, \ldots, X_n$. Suppose that we are interested in finding a valid rate assignment $\mathbf{R}$ and its corresponding power assignment $\mathbf{P}$ for these sources so that the sum power $\sum_{i=1}^n P_i = \sum_{i=1}^n Q_i(R_i)$ is minimum. The optimal valid power assignment can be specified by the minimum weight SMF of G^{tot} which can be found in polynomial time.

IV. Numerical results for noiseless case

Consider a wireless sensor network example in a square area where the x and y coordinates of the sensors are chosen uniformly at random from $[0, 1]$. We use the following entropy model where the individual entropies are assumed to be the same, denoted by H_1 and the joint entropy between two sensors i and j is

$$H(X_i, X_j) = H_1 + (1 - 1/(1 + d_{ij}/c))H_1. \tag{17}$$

where d_{ij} is the distance between i and j and c is a parameter indicating the spatial correlation in the data. Higher c indicates higher correlation.

In Fig. 1, we plot the normalized sum rate R_{s0} vs. the number of sensors n. If there is no pairwise decoding i.e. the nodes transmits data individually to the sink, $R_i = H_1$ and $R_{s0} = n$. The matching solution and the minimum arborescence (MA) solution are compared in the figure. Note that if the nodes are highly correlated $(c = 1)$, the present solution outperforms the matching solution considerably.

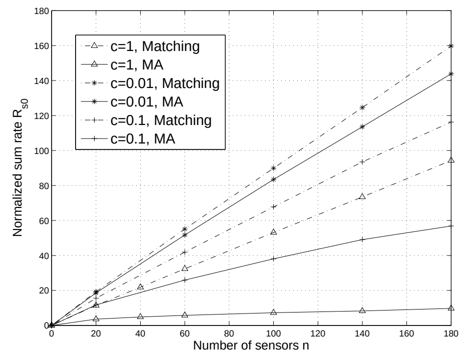

Fig. 1. Normalized sum rate vs. number of sensors

V. Conclusion

We investigated the problems of rate and power allocation for a sensor network where pairwise distributed source coding is used. A more general definition of pairwise distributed source coding was introduced than the one presented in [8]. For the case when the sources and the terminal are connected by noiseless channels, we found a rate allocation with the minimum sum rate. For noisy orthogonal source terminal channels, we found a rate and power allocation with minimum sum power. All algorithms introduced have polynomial-time complexity. Numerical results show that our solution has a significant gain over the solution of [8], especially when correlations are high.

References

[1] D. Slepian and J. Wolf, "Noiseless coding of correlated information sources," *IEEE Trans. on Info. Th.*, vol. 19, pp. 471–480, Jul. 1973.
[2] S. S. Pradhan and K. Ramchandran, "Distributed Source Coding using Syndromes (DISCUS): Design and Construction," *IEEE Trans. on Info. Th.*, vol. 49, pp. 626–643, Mar. 2003.
[3] Z. Xiong, A. D. Liveris, and S. Cheng, "Distributed Source Coding for Sensor Networks," in *IEEE Signal Processing Magazine*, Sept. 2004.
[4] R. Cristescu, B. Beferull-Lozano, and M. Vetterli, "On Network Correlated Data Gathering," in *IEEE Infocom*, 2004.
[5] A. Ramamoorthy, "Minimum cost distributed source coding over a network," in *IEEE Intl. Symposium on Info. Th.*, 2007.
[6] A. Liveris, Z. Xiong, and C. N. Georghiades, "Compression of binary sources with side information at the decoder using LDPC codes," *IEEE Comm. Letters*, vol. 6, no. 10, pp. 440–442, 2002.
[7] A. Liveris, C. Lan, K. Narayanan, Z. Xiong, and C. Georghiades, "Slepian-Wolf coding of three binary sources using LDPC codes," in *Proc. Intl. Symp. on Turbo Codes and Rel. Topics, Brest, France*, Sep. 2003.
[8] A. Roumy and D. Gesbert, "Optimal matching in wireless sensor networks," in *IEEE Intl. Symposium on Info. Th.*, 2007.
[9] J. Kleinberg and E. Tardos, *Algorithm Design*. Addison Wesley, 2005.
[10] B. Rimoldi and C. Urbanke, "Asynchronous Slepian-Wolf coding via source-splitting," in *Proc. Intl. Symp. on Inf. Theory, Ulm, Germany*, Jun.-Jul. 1997, p. 271.
[11] T. P. Coleman and A. H. Lee and M. Medard and M. Effros, "Low-complexity approaches to Slepian-Wolf near-lossless distributed data compression," *IEEE Trans. on Info. Th.*, vol. 52, no. 8, pp. 3546–3561, 2006.
[12] R. Giles, "Optimum matching forests II: General weights," *Mathematical Programming*, vol. 22, no. 1, pp. 12–38, Dec. 1982.