Design of Rate-Compatible Irregular LDPC Codes for Incremental Redundancy Hybrid ARQ Systems

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Abstract— **We present a new class of irregular low-density parity-check (LDPC) codes for finite block length (up to a few thousand symbols). The proposed codes are efficiently encodable and have a simple rate-compatible puncturing structure which is suitable for incremental redundancy hybrid automatic repeat request (IR-HARQ) systems. The codes outperform optimized irregular LDPC codes and (extended) irregular repeat-accumulate codes for rates 0.67~0.94, and are particularly good at high puncturing rates where good puncturing performance has been previously difficult to achieve. These characteristics result in good throughput performance over time-varying channels in IR-HARQ systems.**

I. INTRODUCTION

Many wireless broadband systems require flexible and adaptive transmission techniques since they operate in the presence of time-varying channels. For these systems, incremental redundancy hybrid automatic repeat request (IR-HARQ) schemes are often used, whereby parity bits are sent in an incremental fashion depending on the quality of the time-varying channel [1]. Careful design of an adaptive forward error correction (FEC) code can improve data throughput in such systems. Incremental Redundancy systems require the use of rate-compatible punctured codes (RCPC) [2]. These codes can be operated at different rates by using the same encoder-decoder pair. Depending on the rate requirement, an appropriate number of parity bits are sent by the transmitter. The receiver decodes by treating the parity bits that are not transmitted (called punctured bits) as erasures. In addition, the set of parity bits of a higher rate code forms a subset of the parity bits of a lower rate code. Thus in an IR-HARQ system if the receiver fails to decode at a particular rate it only needs to request additional parity bits from the transmitter.

Low-Density Parity-Check (LDPC) codes are increasingly being considered as good candidates for the next-generation FEC codes in high throughput wireless and recording applications. Their excellent performance and parallelizable decoder make them appropriate for technologies such as DVB-S2, IEEE 802.16e, and IEEE 802.11n. To be useful in IR-HARQ systems LDPC codes need to (a) exhibit good performance under puncturing and (b) have an efficient encoder. Ha *et al.* introduced a good puncturing algorithm with rate-compatible fashion for finite length LDPC codes [3], [4]. This puncturing algorithm shows better puncturing performance than random puncturing for any given LDPC codes. However, the maximum puncturing rate is often limited when this algorithm is applied, so that high puncturing rates are difficult to achieve. Also, designing good mother codes for this puncturing algorithm is a crucial open problem. Efficient encoding of LDPC codes that have block length up to a few thousand bits may be hard unless the code has some algebraic structure. Random constructions of LDPC codes that are known to possess good performance do not have simple encoders in general. Some exceptions are irregular repeat accumulate (IRA) and extended IRA (eIRA) codes [5], [6].

In this work we present a class of efficiently- encodable rate-compatible LDPC codes, called E^2RC codes that have good performance under puncturing as well as efficient encoding scheme [7]. We also present results that demonstrate the superior performance of these codes when used in an IR-HARQ system over time-varying channels. Previous work in IR-HARQ systems includes [8], [9], where the design of an ensemble of FEC codes is considered. IR-HARQ systems require good frame error rate (FER) performance, especially at high rate region to get good throughput performance. Through simulations, we verify that the E^2RC codes have better throughput performance than other comparable LDPC codes since they have good performance at higher puncturing rate.

II. INCREMENTAL REDUNDANCY HYBRID ARQ SYSTEMS

The objective of IR-HARQ scheme is to improve the throughput by retransmitting the required fractional part of the parity bits rather than the whole information and parity bits when the previous transmission fails. The code combining process of our IR-HARQ scheme follows the Chase's rule [10], and details of the steps are as follows:

Code Combining Process for IR-HARQ Scheme

STEP 1: Making a frame with cyclic redundancy check (CRC)
STEP 2: LDPC encoding
STEP 3: Ordering and grouping the parity bits
STEP 4: Transmit the message and/or the required parity group

At the receiver end, the frame is reconstructed with the message and parity groups of the previous frame after receiving the parity group of the current frame. Then, the frame is decoded with LDPC decoder. We detect errors with the help of CRC detection. If errors occur in the current frame, send the negative acknowledgement (NACK) signal to the transmitter, and the transmitter sends the next required parity group. Otherwise,

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sends the acknowledgement (ACK) signal to the transmitter. If the transmitter receives an ACK signal, it stops sending the current frame and prepares the next frame.

An important performance measure of an IR-HARQ scheme is the throughput, which is defined as the ratio of the number of information bits k, to the total number of bits that need to be transmitted for acceptance by the receiver. The throughput, η is given by

$$\eta = \frac{k}{\left(1-F(1)\right)\left(k+p_1\right)+\sum_{i=2}^{\infty}\left(\left(1-F(i)\right)\prod_{j=1}^{i-1}F(j)\right)\left(k+\sum_{j=1}^{i}p_j\right)},$$

where $F(i)$ is the probability of frame error at the i-th transmission, and p_j is the length of parity group at the j-th transmission. In the simulation, we consider k=1024, and p_j's are used as in Table I.

III. The Code Construction Algorithm

Three rules for the design of finite length LDPC codes were proposed in [11]. These are (i) assigning degree-2 nodes to nonsystematic bits, (ii) ensuring that the degree-2 nodes do not form a cycle amongst themselves and (iii) avoiding cycles of length-4 in the Tanner graph of the code. The eIRA codes form the parity part in the bi-diagonal structure which satisfies the above design rules. It is interesting to see whether there exist other ways of placing the degree-2 nodes so that the above conditions are satisfied. We present below an example of such a placement in Fig. 1.

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Fig. 1. An example of cycle-free structure with weight-2 nodes.

Observe that the column degree of each node is 2 and that there does not exist any cycle in this matrix in Fig. 1. We shall see later that this construction can be generalized and the resulting matrices can be used to construct LDPC codes that can be efficiently encoded and have good puncturing performance across a wide range of rates.

A. Rate-Compatible Puncturing Algorithm

For finite length (up to several thousand symbols) LDPC codes, Ha *et al*. proposed an efficient puncturing algorithm to have a rate-compatible code [3], [4]. Since our code construction technique is inspired by it, we present a brief description of the algorithm below.

The puncturing algorithm in [3], [4] is based on the following two definitions. A punctured variable node p is called *1*-step recoverable (*1*-SR) node if it has at least one neighboring check node (called survived check node) whose neighboring variable nodes are all unpunctured except for p. Generalizing this, a punctured variable node p is called k-step recoverable (k-SR) node (see Fig. 2) if it has at least one survived check node that is connected to at least one (k-*1*)-SR node. Furthermore the other nodes that it is connected to need to be m-SR nodes, where $0 \le m \le k-1$. Under these conditions, note that the k-SR node will be recovered after exactly k iterations of iterative decoding assuming that the channel does not cause any errors. So a higher number of low-SR nodes will reduce the overall number of iterations, which results in good puncturing performance. The general idea is to puncture the lower-SR nodes first. By doing so, punctured nodes can be recovered with the help of other unpunctured nodes in lesser number of iterations. The puncturing algorithm is focused on the selection of the nodes to be punctured by maximizing the number of lower-SR nodes for a given parity-check matrix. The important open problems that have not been addressed previously are (i) strategies for increasing the maximum puncturing rate and (ii) the design of good codes for puncturing, i.e. the design of codes that have a large number of low-SR nodes. These are motivations to design the proposed E^2RC codes.

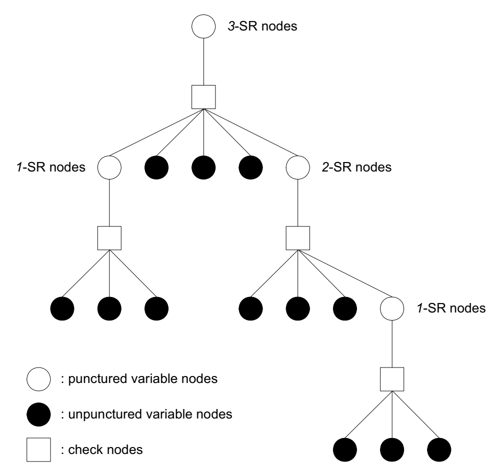

Fig. 2. k-SR node in a graph.

B. Algorithm for E^2RC Code Construction

We consider designing codes that have a large number of lower-SR nodes for good puncturing performance. Before describing our design algorithm, we introduce a few definitions. The matrix P is called a k-SR matrix of a matrix H if the column set of P is the subset of the column set of H, and the column set of P consist of all the k-SR nodes in the matrix H. We define the depth d as the number of k-SR matrices in the matrix H, and $\gamma(k)$ as the number of columns in a k-SR matrix. In our construction we form the non-systematic part of the mother code parity-check matrix by laying the k-SR matrices sequentially as shown in Fig. 3. We regard the last weight-1 column as (d+*1*)-SR matrix.

In this paper, we only describe when M (number of parity bits) is power of two. However, we can apply this algorithm to any M. If M is power of two, we can obtain $d = \log_2 M$, and $\gamma(k) = M / 2^k$. Once we set the design parameters, we generate k-SR matrix of $M \times \gamma(k)$, where $1 \le k \le d$. The j-th column of k-SR matrix has the following sequence:

$$h_{k,j} = D^{\,j+\sum_{i=1}^{k-1}\gamma(i)}\left(1 + D^{\gamma(k)}\right), \quad where \quad 1 \le k \le d,\, 0 \le j \le \gamma(k) - 1$$

$$h_{d+1} = D^{M-1}.$$

In the sequence, D^i represents the position of nonzero element in a column, i.e., i-th element of the column is nonzero, where $0 \le i \le M-1$. Let matrix H_1 and H_2 be the systematic part and non-systematic part of the parity-check matrix H, respectively. Then, we can construct H_2 matrix laying k-SR matrices sequentially as shown in Fig. 3.

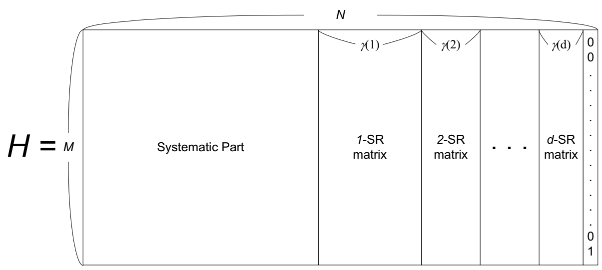

Fig. 3. The parity-check matrix construction of E[2]RC codes.

For a desired code rate, we need to find the optimal degree distribution using density evolution [11]. Note that the degree distribution of the non-systematic part is already fixed by the above generating sequence. From the generating sequence, we can notice that every column in k-SR matrix has weight two except the last column ((d+1)-SR matrix) which has weight one. We can also obtain the right degree distributions of non-systematic part, H_2, as follows (for details of proof, see [7]):

$$\rho(x) = \sum_{i=1}^{d+1} \rho_i x^{i-1},$$

$$where \;\; \rho_i = \frac{i}{2^i \cdot \left(\sum_{j=2}^{d} \frac{j}{2^j} + \frac{d+1}{2^d} \right)} \quad for\; 1 \le i \le d,$$

$$and \;\; \rho_{d+1} = \frac{d+1}{2^d \cdot \left(\sum_{j=2}^{d} \frac{j}{2^j} + \frac{d+1}{2^d} \right)}.$$

From the above degree distributions, we can determine the exact degree distributions for the nonsystematic parts, namely the H_2 matrix. For a desired code rate, we can find optimal degree distributions for the whole code while fixing these degree distributions for H_2. We can then get the degree distributions for the H_1 matrix. For the systematic part, namely the H_1 matrix, the variable nodes are chosen to have degree greater than two.

Since all of the nodes except one node are degree 2 in H_2, the fraction of degree-2 nodes in degree distributions is very high. For a finite length code, the higher portion of degree-2 nodes cause better threshold performance, but a big fraction of degree-2 nodes can result in a small minimum distance, causing a greater probability of decoding errors and higher error floors. To reduce these effects, we can use methods such as those presented in [12]-[15], when we construct the H_1 matrix. Then, we can finally construct the whole parity-check matrix

$$H = \left[H_1 \;\; | \;\; H_2 \right].$$

For the proposed E[2]RC codes, rate-compatibility can be easily obtained by puncturing nodes from left to right in H_2 matrix. Equivalently, we puncture the nodes in the lower-SR matrix first for a desired code rate.

The proposed codes not only have simple rate-compatible puncturing but also an efficient encoding structure. The encoder can be implemented simply with shift-registers, switches, and exclusive-OR operators [16]. Another way to implement the encoder of the proposed E[2]RC codes is by using a simple erasure decoder. Recall that all the nodes in k-SR matrix can be recovered in k iterations with erasure decoder since they are all k-SR nodes. For the proposed codes, even if all the parity bits are erased, we can obtain the exact parity bits within (d+1) iterations using a simple erasure decoder or general LDPC decoder of message-passing algorithm when the channel has no errors. In the transceiver system, this can be a great advantage in terms of complexity. We only need to provide an LDPC decoder for both encoding and decoding, and do not need any extra encoder.

IV. System Model and Simulation Results

A. System Model

As a system model with the IR-HARQ scheme, we consider a LDPC coded Vertical Bell Labs Layered Space-Time (V-BLAST) system in time-varying multiple antenna environments in Fig. 4. The throughput and spectrum efficiency of this system can be improved by using LDPC codes, which are powerful capacity-approaching codes with feasible decoding complexity.

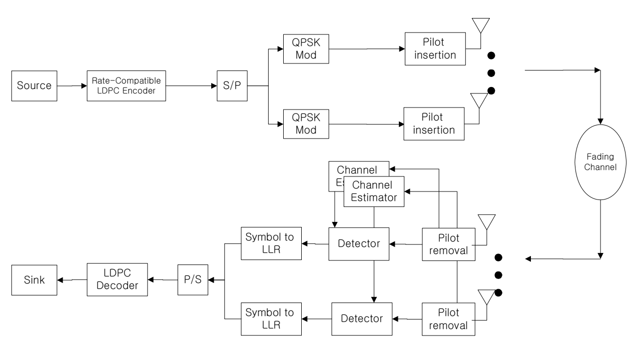

Fig. 4. V-BLAST MIMO system.

The original V-BLAST scheme [17] uses different channel codes in different layers. In this work, we only consider the single LDPC code as a channel code, and separate the output in parallel for each layer. We consider 2×2 MIMO system, which has 2 transmit antennas and 2 receive antennas over the frequency flat Rayleigh fading channel. At the transmitter, the source data bits are encoded with an LDPC encoder, separated into two substreams, and mapped onto quadrature phase shift keying (QPSK) constellation points for each substream. At the receiver side, the received signal can be expressed mathematically as $\boldsymbol{Y}=\boldsymbol{HX}+\boldsymbol{W}$, where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are complex input and output vectors, respectively, and $\boldsymbol{W}$ is a complex Gaussian noise with a variance σ^2. The complex 2×2 channel matrix is $\boldsymbol{H}$, which consists of channel coefficients of MIMO frequency-flat fading channels. At the receiver, perfect channel estimation is assumed, and the minimum mean square error (MMSE) detector is used for making a soft decision on the channel inputs. Then, each received soft bit stream is multiplexed into one stream and converted into a stream of log-likelihood ratio (LLR) values. These are used for soft decoding of a log-domain LDPC decoder.

B. *Simulation Results*

We consider rate-1/2 codes with code length of 2048. For IR-HARQ systems, IR parity bits are assigned as in Table I, which are used as subset codes of an ensemble. We assume that the first transmission starts from rate of 0.94. We compare the FER and throughput performance of E^{2}RC codes with those of eIRA codes and general irregular LDPC codes.

TABLE I. ENSEMBLE OF LDPC CODES IN THE IR-HARQ SIMULATION

i	1	2	3	4	5	6	7	8	9
p_i	64	64	128	128	128	128	128	128	128
rate	0.94	0.89	0.80	0.73	0.67	0.62	0.57	0.53	0.50

When we generate eIRA codes and general LDPC codes, we try to keep the same degree distributions as those in [6] for rate-1/2 codes, which are optimized in additive white Gaussian noise (AWGN) channel:

$$\lambda(x)=0.00015+0.30235x+0.27132x^2+0.42618x^6$$

$$\rho(x)=0.35555x^5+0.64445x^6.$$

For E^{2}RC codes, however, the actual degree distributions are slightly different to compensate the right degree of H_2.

$$\lambda(x)=0.00015+0.30235x+0.27132x^2+0.42618x^6$$

$$\rho(x)=0.41140x^5+0.54617x^6+0.01892x^7+0.01064x^8$$

$$+0.00592x^9+0.00325x^{10}+0.00178x^{11}+0.00193x^{12}.$$

We apply the progressive edge growth (PEG) algorithm proposed in [12] to H_1 design of eIRA codes and E^{2}RC codes for having the better girth characteristics. First, we compare the puncturing performance between the proposed E^{2}RC codes and the eIRA codes. We apply the intentional puncturing algorithm proposed in [3], [4] to the eIRA codes, and compare the FER performance with E^{2}RC codes (see Fig. 5). In this case, we face puncturing limitations. In fact, the puncturing algorithm in [3], [4] assigns 512 nodes as *1*-SR nodes and cannot find any more *k*-SR nodes ($k\geq2$) if we try to maximize the number of *1*-SR nodes. To get a high rate in eIRA codes we puncture randomly after the puncturing limitation (512 *1*-SR nodes). This destroys the previous tree structure of *1*-SR nodes resulting in poor performance. The puncturing performance of the E^{2}RC codes is better than that of eIRA codes as the code rates are increased even though we apply the best effort puncturing algorithm to eIRA codes.

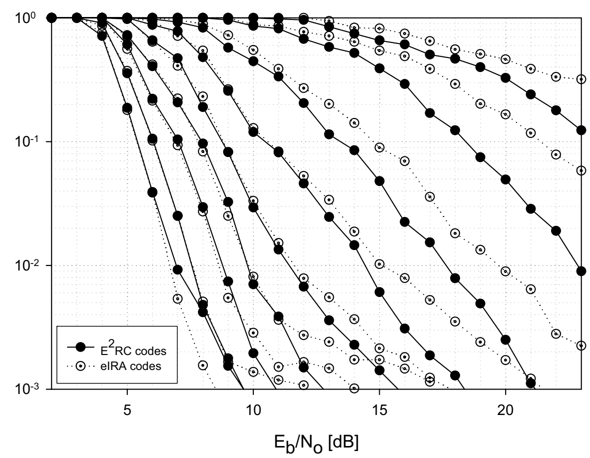

Fig. 5. Performance comparison of rate-1/2 E^{2}RC codes (filled circle) and eIRA codes (unfilled circle). The message size is 1024 bits and curves are for rate = 0.5, 0.53, 0.57, 0.62, 0.67, 0.73, 0.80, 0.89, 0.94 from left to right.

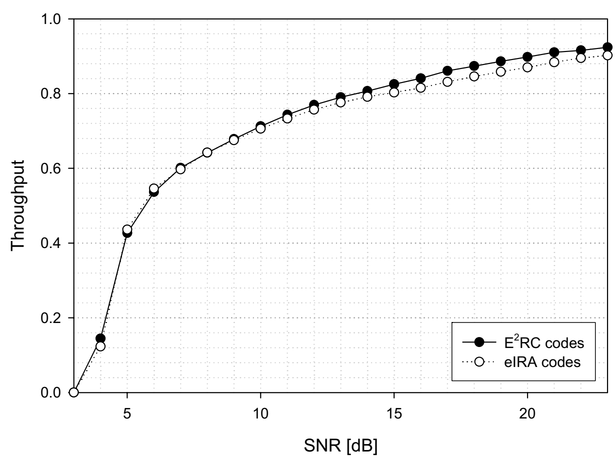

Fig. 6. Throughput performance comparison of E^{2}RC codes (filled circle) and eIRA codes (unfilled circle). The message size is 1024 bits for both codes.

For throughput simulations, we consider FER of 10^{-3}, and simulate codes over the IR-HARQ scheme presented in section II. We present the throughput performance comparison between E^{2}RC and eIRA codes in Fig. 6. At the throughput of 0.8 in Fig. 6, the E^{2}RC codes have 2dB gain over eIRA codes. This is because as mentioned earlier the throughput performance highly depends on the high puncturing rate.

To compare the performance with general irregular LDPC codes, we also apply the PEG algorithm in [12] to generate the code. From the rate-1/2 mother codes, we provide punctured codes of rate as following the Table I using the puncturing algorithm in [3], [4]. Through the simulation, we observe that the FER performance of E^{2}RC codes is slightly worse than or equal to general LDPC codes at lower code rate (rates 0.5~0.62), but outperforms them at higher code rate (rates 0.67~0.94). For this reason, the E^{2}RC codes show better throughput performance than irregular LDPC codes as shown in Fig. 7. The E^{2}RC codes have a gain of about 2.2 dB at the throughput of 0.8.

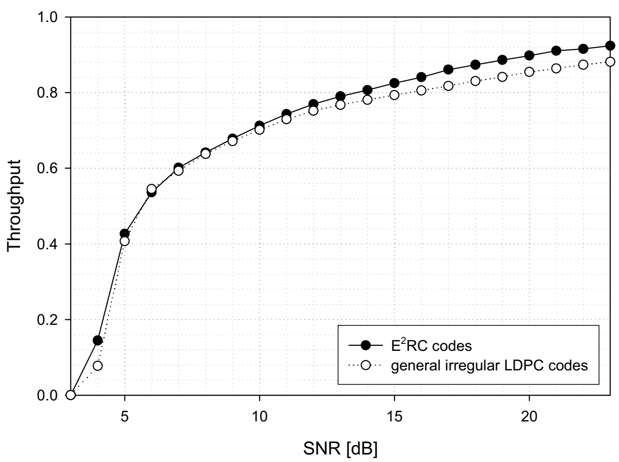

Fig. 7. Throughput performance comparison of E^{2}RC codes (filled circle) and general irregular LDPC codes (unfilled circle). The message size is 1024 bits for both codes.

V. Conclusion

We have proposed a class of codes, E^{2}RC codes that have several strong points. First, they are efficiently encodable, which enables low-complexity encoder implementation. We also showed that a simple erasure decoder can also be used for the linear-time encoding of these codes. Thus, we can share a message-passing decoder for both encoding and decoding if it is applied to the transceiver systems which require an encoder/decoder pair. Second, E^{2}RC codes are suitable for rate-compatible puncturing. The E^{2}RC codes show better puncturing performance than other irregular LDPC codes and eIRA codes in all ranges of code rates, especially in high puncturing rate. These characteristics result in good threshold performance over time-varying channel in IR-HARQ systems. From simulations we observe that E^{2}RC codes outperform eIRA codes and general irregular LDPC codes by 2dB and 2.2dB, respectively, at the throughput of 0.8.

Acknowledgement

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