Formulae Relating Controllability, Observability, and Co-Observability

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Abstract

We present formulae relating controllability, observability, and co-observability arising in the context of supervisory control of discrete event systems. Given a discrete event plant $G$ with event set $\Sigma$, uncontrollable event set $\Sigma_{ui}$, and observation mask $M_i$ of the $i$th supervisor, we first show that the infimal prefix-closed, $(L(G), \bigcap_i \Sigma_{ui})$-controllable, and $(L(G), \Sigma_{ui}, M_i)$–co-observable superlanguage equals the intersection (taken over all $i$’s) of the infimal prefix-closed, $(L(G), \Sigma_{ui})$-controllable, and $(L(G), \Sigma_{ui}, M_i)$-observable superlanguage. Next we show that the infimal prefix-closed, and $(\Sigma^*, M)$-observable superlanguage computation preserves $(\Sigma^*, \Sigma_{ui})$-controllability. These results can be used to compute individual supervisors in the decentralized control setting.

Key words: Discrete-event systems. Supervisory Control. Controllability. Observability. Centralized Control. Decentralized Control.

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1 Introduction

Ramadge-Wonham (1989) in their pioneering work on supervisory control of discrete event systems (DESs) introduced the notion of controllability as a necessary and sufficient condition for the existence of a supervisor that achieves a desired controlled behavior of a given discrete event plant under the complete observation of events. However, if the events are not completely observed by the supervisor, rather its observation is filtered by an observation mask, then the additional condition of observability introduced by Lin-Wonham (1988), and Cieslak et al. (1988) is needed for its existence. In the more general case of decentralized supervision, more than one supervisors control the plant such that the uncontrollable event set for the $i$th supervisor is $\Sigma_{ui}$, and its observation is filtered through the mask $M_i$. In this setting, the existence condition of co-observability was introduced by Cieslak et al. (1988) for the case when controlled behavior is given as a prefix-closed language, and later generalized to the non-prefix-closed case by Rudie-Wonham (1992).

In this note we first obtain a relationship between controllability, observability, and co-observability. Given a plant $G$ with event set $\Sigma$, uncontrollable event sets $\Sigma_{ui}$, and observation masks $M_i$ for the $i$th supervisor, we prove that the infimal prefix-closed, $(L(G), \cap_i \Sigma_{ui})$-controllable and $(L(G), \Sigma_{ui}, M_i)$-co-observable superlanguage of the specification language equals the intersection (taken over all $i$'s) of infimal prefix-closed $(L(G), \Sigma_{ui})$-controllable and $(L(G), M_i)$-observable superlanguages. This result can be used to obtain the individual supervisors for decentralized control, where the $i$th supervisor generates the infimal prefix-closed, $(L(G), \Sigma_{ui})$-controllable, and $(L(G), M_i)$-observable superlanguage of the given specification.

The computation of individual supervisors in the decentralized control setting requires the computation of the infimal prefix-closed, $(\Sigma^*, \Sigma_u)$-controllable, and $(\Sigma^*, M)$-observable superlanguage, which we show can be computed modularly: First compute the infimal prefix-closed and $(\Sigma^*, \Sigma_u)$-controllable superlanguage and then compute its infimal prefix-closed and $(\Sigma^*, M)$-observable superlanguage. This result is a consequence of the fact that the infimal prefix-closed, and $(\Sigma^*, M)$-observable superlanguage computation preserves $(\Sigma^*, \Sigma_u)$-controllability. In the special case when the observation mask is a natural projection, a “dual” result was obtained by Lin-Wonham (1988) (cf. Lemma 3.3) where it was shown that the infimal prefix-closed, and controllable superlanguage computation preserves observability. However, this property does not hold in the more general case of non-projection type observation masks as can be seen from Example 9 below. In contrast the property we present holds for general observation masks and facilitates a “modular” computation of the infimal prefix-closed, controllable, and observable superlanguage.
2 Notation and Preliminaries

\( \Sigma \) is used to denote the universe of events for DESs. \( \Sigma^* \) denotes the set of all finite sequences of events including the zero length sequence denoted as \( \epsilon \). Elements of \( \Sigma^* \) are called strings or traces, and subsets of \( \Sigma^* \) are called languages. For a language \( H \), \( pr(H) \) denotes the set of all prefixes of traces from \( H \). \( H \) is called prefix-closed if \( pr(H) = H \), and the supremal prefix-closed sublanguage of \( H \) is denoted as \( sup P(H) \).

State machines are used for modeling discrete event plants. A nondeterministic state machine (NSM) \( G \) is a five-tuple as defined in Hopcroft-Ullman (1979):

\[ G := (X, \Sigma, \alpha, x_0, X_m), \]

where \( X \) is the state set, \( \alpha : X \times (\Sigma \cup \{ \epsilon \}) \to 2^X \) is the nondeterministic transition function, \( x_0 \in X \) is the initial state, and \( X_m \subseteq X \) is the set of marked or accepting states. A triple \((x_1, \sigma, x_2)\) such that \( x_2 \in \alpha(x_1, \sigma) \) is called a transition; it is called an \( \epsilon \)-transition if \( \sigma = \epsilon \). \( G \) is said to be deterministic if it contains no \( \epsilon \)-transition and \( |\alpha(x, \sigma)| \leq 1 \) for each \( x \in X \) and \( \sigma \in \Sigma \). The transition function is extended to \( \alpha : X \times \Sigma^* \to 2^X \) in the standard way. We use \( L_m(G) \) and \( L(G) \) to denote the marked or accepted and generated language, respectively, of \( G \).

For supervisory control of deterministic systems, a discrete event plant can be represented as a deterministic state machine \( G \) as discussed in Kumar-Garg (1995). The event set of \( G \) is partitioned into \( \Sigma = \Sigma_u \cup (\Sigma - \Sigma_u) \), the sets of uncontrollable and controllable events. Supposing event observations of a supervisor are filtered through an observation mask \( M \); a supervisor is defined to be a map \( f : M(L(G)) \to 2^{\Sigma - \Sigma_u} \) such that for each trace \( s \in L(G) \) executed by the plant, \( f(M(s)) \subseteq \Sigma - \Sigma_u \) is the set of controllable events disabled following \( s \). It is well known that a condition for the existence of a supervisor for the controlled generated language to equal a given specification language is controllability and observability.

Given a prefix-closed language \( H \), \( K \) is said to be \((H, \Sigma_u)\)-controllable if

\[ pr(K)\Sigma_u \cap H \subseteq pr(K); \]

and it is said to be \((H, M)\)-observable if

\[ \forall s, t \in pr(K), \sigma \in \Sigma : M(s) = M(t), s\sigma \in pr(K), t\sigma \in H \Rightarrow t\sigma \in pr(K). \]

The following notation is used for various kinds of infimal prefix-closed superlanguages of \( K \): \( \inf \overline{PC}_{\Sigma_u}(K) \) for \((L(G), \Sigma_u)\)-controllable; \( \inf \overline{PO}_{M}(K) \)
for \((L(G), M)\)-observable; \(\inf \overset{\Sigma_u}{\text{PC}} \overset{M}{O}(K)\) for \((L(G), \Sigma_u)\)-controllable, and
\((L(G), M)\)-observable. Then Lafortune-Chen (1990) show that \(\inf \overset{\Sigma_u}{\text{PC}}(K) = \text{pr}(K) \Sigma_u^* \cap L(G)\), and a formula for \(\inf \overset{M}{\text{PO}}(K)\) was given by Rudie-Wonham (1990) for the special case of projection type observation masks, which can be extended to the general case of non-projection type observation masks by Kumar (1993) and can be given as:

\[
\inf \overset{M}{\text{PO}}(K) = \sup P[\tilde{M}^{-1} \tilde{M}(\text{pr}(K))] \cap L(G),
\]

where \(\tilde{M}\), first introduced by Rudie-Wonham (1990), is defined inductively as:

\[
\tilde{M}(\epsilon) := \epsilon; \quad \tilde{M}(s\sigma) := M(s)\sigma,
\]

i.e., \(\tilde{M}\) “masks” all but the last event.

Also, we use the following notation for another set of infimal prefix-closed superlanguages of \(K\): \(K^{\Sigma_u}\) for \((\Sigma^*, \Sigma_u)\)-controllable; \(K^M\) for \((\Sigma^*, M)\)-observable; and \(K^{\Sigma_u, M}\) for \((\Sigma^*, \Sigma_u)\)-controllable and \((\Sigma^*, M)\)-observable. Then it follows from above that \(K^{\Sigma_u} = \text{pr}(K) \Sigma_u^*\); \(K^M = \sup P[\tilde{M}^{-1} \tilde{M}(\text{pr}(K))]\); and we show below that \(K^{\Sigma_u, M} = (K^{\Sigma_u})^M\).

3 Formulae for Controllability and (Co)-observability

In many situations due to the physically distributed nature of the plant, it is impractical to design a centralized supervisor, and instead our interest is to synthesize two or more supervisors, each having its own observation mask and own set of controllable events, such that a desired controlled behavior is achieved. Let \(\Sigma_{ui}\) denote the uncontrollable event set and \(M_i\) denote the observation mask for the \(i\)th supervisor. Without loss of generality, we consider “two-decentralization”, i.e., \(i = 1, 2\). Then \(\Sigma_u := \Sigma_{u1} \cap \Sigma_{u2}\) represents the set of events that are uncontrollable to both of the supervisors. So for the existence of decentralized supervision it is expected that the desired behavior be \((L(G), \Sigma_u)\)-controllable. On the other hand, the events in the set \(\Sigma - \Sigma_u\) can be controlled by at least one of the supervisors. However, their enablement and disablement must satisfy the restriction imposed by the partial observability of the supervisors. This leads to the introduction of the notion of co-observability which together with controllability is necessary and sufficient for decentralized supervision as shown by Rudie-Wonham (1992):

**Definition 1** Given a prefix-closed language \(H\), and the uncontrollable event sets \(\Sigma_{ui}\) and the observation masks \(M_i\) \((i = 1, 2)\), \(K\) is said to be \((H, \Sigma_{ui}, M_i)\)-
There are different cases of the definition of co-observability. First suppose \( \sigma \in \Sigma_{u2} - \Sigma_{u1} \), \( M_1(s_1) = M_1(t), s_1\sigma \in pr(K), t\sigma \in H \Rightarrow t\sigma \in pr(K) \)

(2) \( \sigma \in \Sigma_{u1} - \Sigma_{u2} \), \( M_2(s_2) = M_2(t), s_2\sigma \in pr(K), t\sigma \in H \Rightarrow t\sigma \in pr(K) \)

(3) \( \sigma \in (\Sigma_{u1} \cup \Sigma_{u2})^c \), \( M_1(s_1) = M_1(t), M_2(s_2) = M_2(t), s_1\sigma, s_2\sigma \in pr(K), t\sigma \in H \Rightarrow t\sigma \in pr(K) \),

where \( (\Sigma_{u1} \cup \Sigma_{u2})^c := \Sigma - (\Sigma_{u1} \cup \Sigma_{u2}) \).

A test for co-observability is given by Rudie-Willems (1995). We need the result of the following lemma before we present a formula relating controllability, observability, and co-observability.

**Lemma 2** Consider a plant \( G \), event sets \( \Sigma_{u1}, \Sigma_{u2} \) with \( \Sigma_u := \Sigma_{u1} \cap \Sigma_{u2} \), and masks \( M_1, M_2 \). If for \( i = 1, 2 \), \( K_i \subseteq \Sigma^* \) is prefix-closed, \( (L(G), \Sigma_{u1}) \)-controllable and \( (L(G), M_i) \)-observable, then \( K := K_1 \cap K_2 \) is prefix-closed, \( (L(G), \Sigma_u) \)-controllable and \( (L(G), \Sigma_{ui}, M_i) \)-co-observable.

**PROOF.** Prefix-closure of \( K \) follows from prefix-closure of each \( K_i \). Since \( K_i \) is \( (L(G), \Sigma_{ui}) \)-controllable and \( \Sigma_u \subseteq \Sigma_{ui} \), it follows that \( K_i \) is \( (L(G), \Sigma_u) \)-controllable. Since \( K_i \) is prefix-closed, and controllability of prefix-closed languages is preserved under intersection, it follows that \( K \) is \( (L(G), \Sigma_u) \)-controllable.

In order to show co-observability, pick \( s_1, s_2, t \in K \) (note that \( K \) is prefix-closed, so \( pr(K) = K \)) and \( \sigma \in \Sigma - \Sigma_u \). Then we must consider the three different cases of the definition of co-observability. First suppose \( \sigma \in \Sigma_{u2} - \Sigma_{u1}, M_1(s_1) = M_1(t), s_1\sigma \in K \) and \( t\sigma \in L(G) \); we need to show that \( t\sigma \in K \). Since \( t \in K_i \) (as \( t \in K \)), it follows from the \( (L(G), M_i) \)-observability of \( K_i \) that \( t\sigma \in K_i \). On the other hand, since \( t \in K_2 \) (as \( t \in K \)) and \( \sigma \in \Sigma_{u2} - \Sigma_{u1} \subseteq \Sigma_{u2}, \) it follows from the \( (L(G), \Sigma_{u2}) \)-controllability of \( K_2 \) that \( t\sigma \in K_2 \). The second case of co-observability can be obtained symmetrically, and the final case follows from the \( (L(G), M_i) \)-observability of \( K_i \). \( \square \)

The result of Lemma 2 can be used to obtain the main result of this note. For notational simplicity, define \( K^i := \inf \mathcal{P}C_{\Sigma_{ui}} O_{M_i}(K) \) and let \( K^{12} \) be the infimal prefix-closed, \( (L(G), \Sigma_u) \)-controllable, and \( (L(G), \Sigma_{ui}, M_i) \)-co-observable superlanguage of \( K \).

\(^2\) We have used this concise terminology to mean \( (H, \Sigma_{u1}, \Sigma_{u2}, M_1, M_2) \)-co-observable, and it is not to be confused with \( (H, \Sigma_{u1}, M_1) \)-co-observable and \( (H, \Sigma_{u2}, M_2) \)-co-observable.
**Theorem 3** Consider a plant $G$ and desired behavior $K$. Let $\Sigma_{ui}, M_i, K^i (i = 1, 2)$, and $K^{12}$ be as defined above. Then $K^{12} = K^1 \cap K^2$.

**PROOF.** Clearly, the required equality holds when $K = \emptyset$. So we assume that $K \neq \emptyset$.

By definition $K \subseteq K^i$; hence that $K \subseteq K^1 \cap K^2$. Also, it follows from Lemma 2 that $K^1 \cap K^2$ is prefix-closed, $(L(G), \Sigma_u)$-controllable, and $(L(G), \Sigma_{ui}, M_i)$-co-observable. So $K^1 \cap K^2$ is a prefix-closed, $(L(G), \Sigma_u)$-controllable and $(L(G), \Sigma_{ui}, M_i)$-co-observable superlanguage of $K$. Since $K^{12}$ is the infimal such language, we have that $K^{12} \subseteq K^1 \cap K^2$.

Next, to see the reverse containment, it suffices to show that non-zero length strings of $K^1 \cap K^2$ are also in $K^{12}$, as the zero length string $\epsilon$ does belong to $K^{12}$ (recall that $K \neq \emptyset$). Thus we need to show that for any string $t \in K^{12}$ and an event $\sigma$ such that $t\sigma \in K^1 \cap K^2$, $t\sigma \in K^{12}$. If $\sigma \in \Sigma_u$, then it follows from the prefix-closure, and $(L(G), \Sigma_u)$-controllability of $K^2$ that $t\sigma \in K^{12}$.

(However, by definition $K^1, K^2 \subseteq L(G)$, which implies $t\sigma \in L(G)$.)

On the other hand, if $\sigma \in \Sigma - \Sigma_u$, then it suffices to show that the following holds:

\begin{align*}
(1) & [\sigma \in \Sigma_{u2} - \Sigma_{u1}] \Rightarrow [\exists s_1 : M_1(s_1) = M_1(t), s_1\sigma \in pr(K)] \\
(2) & [\sigma \in \Sigma_{u1} - \Sigma_{u2}] \Rightarrow [\exists s_2 : M_2(s_2) = M_2(t), s_2\sigma \in pr(K)] \\
(3) & [\sigma \in (\Sigma_{u1} \cup \Sigma_{u2})^\ast] \Rightarrow [\exists s_1, s_2 : M_1(s_1) = M_1(t), M_2(s_2) = M_2(t), s_1\sigma, s_2\sigma \in pr(K)],
\end{align*}

as this together with co-observability of $K^{12}$ clearly implies that $t\sigma \in K^{12}$.

We only prove that case (1) holds, as the other two cases can be proved analogously. Since $t\sigma \in K^1 \cap K^2 \subseteq K^1 \subseteq L(G)$, $t\sigma \in K^1$ and $t\sigma \in L(G)$. Also, since $\sigma \in \Sigma_{u2} - \Sigma_{u1}$, $\sigma \notin \Sigma_{u1}$. Since $K^1$ is the infimal prefix-closed $(L(G), \Sigma_{u1})$-controllable, and $(L(G), M_1)$-observable superlanguage of $K$, and since $\sigma \notin \Sigma_{u1}$, there exists $s_1 \in K^1$ such that $M_1(s_1) = M_1(t)$ and $s_1\sigma \in pr(K)$, as desired. (If such a trace $s_1 \in K^1$ does not exist, i.e., for all $s_1 \in K^1$ with $M_1(s_1) = M_1(t)$ implies $s_1\sigma \notin pr(K)$, then this together with observability of $K^1$ implies $s_1\sigma \in K^1 - pr(K)$. So the language $K^1 - \{s_1\sigma\Sigma^* \mid s_1 \in K^1, M_1(s_1) = M_1(t)\}$ obtained by disabling $\sigma$ after each trace in $K^1$ that is indistinguishable from the trace $t$, is a prefix-closed, $(L(G), \Sigma_{u1})$-controllable and $(L(G), M_1)$-observable superlanguage of $K$ that is strictly contained in $K^1$, a contradiction.) \(\square\)

**Remark 4** Theorem 3 establishes the desired relationship between controllability, observability, and co-observability. It suggests that if it is desired to achieve the infimal prefix-closed, $(L(G), \cap, \Sigma_{ui})$-controllable, and $(L(G), \Sigma_{ui}, M_i)$-co-observable superlanguage of the given specification language as the con-
trolled generated language using decentralized control, then the individual supervisors can be constructed such that the $i$th supervisor generates the infimal prefix-closed, $(L(G), \Sigma_{ui})$-controllable, and $(L(G), M_i)$-observable superlanguage of the given specification language.

Theorem 3 can be used to obtain the following corollary which provides an alternate characterization of the existence of decentralized control.

**Corollary 5** Consider the plant $G$ and the desired behavior $K$. Let $\Sigma_{ui}, M_i, K^i$ $(i = 1, 2)$ be as defined above. Then $K$ is $(L(G), \Sigma_u)$-controllable and $(L(G), \Sigma_{ui}, M_i)$-co-observable if and only if $K^1 \cap K^2 \subseteq pr(K)$.

**PROOF.** We have the following series of equivalences:

\[
K \text{ is } (L(G), \Sigma_u)-\text{controllable , } (L(G), \Sigma_{ui}, M_i)-\text{co-observable} \\
\Leftrightarrow K^{12} = pr(K) \\
\Leftrightarrow K^1 \cap K^2 = pr(K) \\
\Leftrightarrow K^1 \cap K^2 \subseteq pr(K),
\]

where the second equivalence follows from Theorem 3 and the final equivalence follows from the fact that the containment $pr(K) \subseteq K^1 \cap K^2$ always holds. \(\square\)

We conclude this section by providing a result which facilitates “modular” computation of individual supervisors in the decentralized control setting. As stated in Remark 4, the $i$th supervisor in the decentralized setting generates the language $K^i$. We show in the next lemma that $K^i = K^{\Sigma_{ui}, M_i} \cap L(G)$, and in the next theorem that $K^{\Sigma_{ui}, M_i} = (K^{\Sigma_u})^{M_i}$; these together suggest a “modular” construction of the $i$th supervisor.

**Lemma 6** Consider a plant $G$ and a desired behavior $K$. For $i = 1, 2$, let $K^i$ and $K^{\Sigma_{ui}, M_i}$ be as defined above. Then $K^i = K^{\Sigma_{ui}, M_i} \cap L(G)$.

**PROOF.** Clearly, the result holds for $K = \emptyset$. So we assume $K \neq \emptyset$.

It is easy to verify that $K^{\Sigma_{ui}, M_i} \cap L(G)$ is prefix-closed, $(L(G), \Sigma_{ui})$-controllable, and $(L(G), M_i)$-observable superlanguage of $K$. Since $K^i$ is the infimal such superlanguage, the forward containment follows.

To see the reverse containment, it suffices to show that non-zero length traces of $K^{\Sigma_{ui}, M_i} \cap L(G)$ are in $K^i$, as the zero length trace does belong to $K^i$ (recall that $K \neq \emptyset$). Thus we need to show for any $t \in K^i$ and $\sigma \in \Sigma$...
such that \( t\sigma \in K^{\Sigma \cup i,M_i} \cap L(G) \), \( t\sigma \in K^i \). It follows from the \((L(G), \Sigma u_i)\)-controllability of \( K^i \) that this clearly holds for any \( \sigma \in \Sigma u_i \). On the other hand, if \( \sigma \notin \Sigma u_i \), then as in the proof of Theorem 3, there exists \( s_i \in K^{\Sigma u_i,M_i} \) such that \( M_i(s_i) = M_i(t) \) and \( s_i\sigma \in \text{pr}(K) \subseteq K^i \). So from the \((L(G), M_i)\)-observability of \( K^i \) it follows that \( t\sigma \in K^i \), as desired. \( \square \)

It only remains to provide a “modular” computation of \( K^{\Sigma u_i,M_i} \) for each \( i \). For simplicity, we drop the index \( i \) and provide the desired result for modular computation of \( K^{\Sigma u,M} \) with the obvious connotation.

**Theorem 7** The following holds for a language \( K \): \( K^{\Sigma u,M} = (K^{\Sigma u})^M \).

**PROOF.** In order to show the forward containment it suffices to show that \((K^{\Sigma u})^M \) is a prefix-closed, \((\Sigma^*, \Sigma u)\)-controllable, and \((\Sigma^*, M)\)-observable superlanguage of \( K \) since \( K^{\Sigma u,M} \) is the infimal such superlanguage. By definition, \((K^{\Sigma u})^M \) is a prefix-closed, and \((\Sigma^*, M)\)-observable superlanguage of \( K \); and it remains to show that it is also \((\Sigma^*, \Sigma u)\)-controllable, i.e., the operation \((\cdot)^M \) preserves \((\Sigma^*, \Sigma u)\)-controllability. Pick \( s \in (K^{\Sigma u})^M \) and \( \sigma \in \Sigma u \). We need to show that \( s\sigma \in (K^{\Sigma u})^M \). We first prove by induction that there exists \( t \in \Sigma^* \) such that \( M(t) = M(s) \) and \( t\sigma \in K^{\Sigma u} \). Since \( \sigma \in \Sigma u \), clearly this holds for \( s = \epsilon \) by choosing \( t = \epsilon \), which proves the base step. For the induction step suppose \( s = \overline{\sigma} \tau \) with \( \overline{\sigma}, \tau \in \Sigma \), then from induction hypothesis there exists \( \overline{t} \in \Sigma^* \) such that \( M(\overline{t}) = M(\overline{\sigma}) \) and \( \overline{t}\sigma \in K^{\Sigma u} \). Since \( \sigma \in \Sigma u \), this implies \( \overline{t}\sigma \sigma \in K^{\Sigma u} \subseteq (K^{\Sigma u})^M \). Finally, since \( M(\overline{t}\sigma) = M(\overline{\sigma}) \), we must have \( \overline{\sigma} \tau \sigma = s\sigma \in (K^{\Sigma u})^M \), which establishes the induction step.

Next, by definitions of \( K^{\Sigma u} \) and \( K^{\Sigma u,M} \) we have \( K^{\Sigma u} \subseteq K^{\Sigma u,M} \). Hence \((K^{\Sigma u})^M \subseteq (K^{\Sigma u,M})^M = K^{\Sigma u,M} \), which proves the backward containment. \( \square \)

**Remark 8** As a final remark we note that the modular computation suggested by Theorem 7 does not hold if controllability and observability are taken with respect to \( L(G) \) (rather than with respect to \( \Sigma^* \)), i.e., the infimal prefix-closed, and \((L(G), M)\)-observable superlanguage computation does not preserve the \((L(G), \Sigma u)\)-controllability. This is illustrated in the example below.

We also note that a result “dual” to Theorem 7 was first presented by Lin-Wonham (1988), (cf. Lemma 3.3), where it was shown that in the special case when the observation mask is projection type, the infimal prefix-closed, and \((L(G), \Sigma u)\)-controllable superlanguage computation preserves \((L(G), M)\)-observability. However, in contrast to the result of Theorem 7 which holds for general observation masks, the result of Lemma 3.3 of Lin-Wonham (1988) does not hold when the observation mask is non-projection type as demonstrated by the simple example below.
**Example 9** Suppose $\Sigma = \{a, b\}$, $\Sigma_u = \{b\}$, $M(a) = M(b) \neq \epsilon$. Let $L(G) = \text{pr}(\{aab, ba\})$ and $K = \text{pr}(\{a, ba\})$. Then clearly, $K$ is $(L(G), \Sigma_u)$-controllable. However, since $a, b \in K$, $M(a) = M(b)$, $ba \in K$, and $aa \in L(G) - K$, $K$ is not $(L(G), M)$-observable. The infimal prefix-closed, $(L(G), M)$-observable super-language of $K$ can be seen to be $\text{pr}(\{aa, ba\})$. This however, is not $(L(G), \Sigma_u)$-controllable validating the first claim of the previous remark.

Next to validate the second claim of the remark consider $L(G) = \text{pr}(\{aaa, baa, bba\})$, and $K = \text{pr}(\{aaa, baa\})$. Then it is easy to see that $K$ is $(L(G), M)$-observable. But since $b \in K$ and $bb \in L(G) - K$ with $b \in \Sigma_u$, it follows that $K$ is not $(L(G), \Sigma_u)$-controllable. The infimal prefix-closed, and controllable super-language of $K$ can be easily computed to be $\text{pr}(\{aaa, baa, bb\})$. This however, is not $(L(G), M)$-observable since $ba, bb \in K$, $M(bb) = M(ba)$, $baa \in K$, but $bba \in L(G) - K$. Thus the infimal prefix-closed, and controllable computation does not preserve observability in the general case when the observation mask is non-projection type.

### 4 Conclusion

We have presented a formula relating controllability, observability, and co-observability that suggests a technique for the construction of individual supervisors in the setting of decentralized control. We further show that the construction of the individual supervisors can be modularized by showing that the infimal prefix-closed, and $(\Sigma^*, M)$-observable superlanguage computation preserves $(\Sigma^*, \Sigma_u)$-controllability.

### References


