is equivalent to

\[ x = 0 \text{ is unstable} \iff z = 0 \text{ is unstable} \]

- 4.27 (a) The equilibrium points are the roots of the equations

\[
0 = -x_2x_3 + 1, \quad 0 = x_1x_3 - x_2, \quad 0 = x_3^2(1 - x_3)
\]

From the third equation, \( x_3 = 0 \) or \( x_3 = 1 \). The first equation cannot be satisfied with \( x_3 = 0 \).

\[
x_3 = 1 \Rightarrow x_2 = 1 \Rightarrow x_1 = x_2 = 1
\]
Hence, there is a unique equilibrium point at (1, 1, 1).

(b)\[
\begin{bmatrix}
0 & -x_3 & -x_2 \\
x_3 & -1 & x_1 \\
0 & 0 & 2x_3 - 3x_3^2 
\end{bmatrix}_{x=(1,1,1)} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}
\]

The eigenvalues are $-1$ and $(-1 \pm j\sqrt{3})/2$. Hence, the origin is asymptotically stable. The third state equation has equilibrium at $x_3 = 0$. Starting with the initial condition $x_3(0) = 0$, we have $x_3(t) \equiv 0$. Then, $\dot{x}_1 = 1$ and $x_1(t)$ grows unbounded. Thus, the equilibrium point is not globally asymptotically stable.

- 4.28

(a)\[
0 = x_1, \quad 0 = (x_1x_2 - 1)x_2^3 + (x_1^2 - 1 + x_1^2)x_2
\]

Substitution of $x_1 = 0$ in the second equation yields\[
-x_2(1 + x_2^2) = 0 \Rightarrow x_2 = 0
\]

Hence, the origin is the unique equilibrium point.

(b)\[
\begin{bmatrix}
0 & -1 & 0 \\
-1 & 0 & -1 \\
0 & -1 & 0
\end{bmatrix}_{x=0} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}
\]

Hence, the origin is asymptotically stable.

(c) Let $V(x) = x_1x_2$.

\[
\dot{V}(x) = x_1\dot{x}_2 + \dot{x}_1x_2 = (x_1x_2 - 1)x_1x_2^3 + (x_1^2 - 1 + x_1^2)x_1x_2 - x_1x_2
\]

which implies that $\Gamma$ is a positively invariant set.

(d) The origin is not globally asymptotically stable since trajectories starting in $\Gamma$ do not converge to the origin.

- 4.29 (a) The equilibrium points are the roots of the equations\[
0 = x_1 - x_1^3 + x_2, \quad 0 = 3x_1 - x_2
\]

The equilibrium points are $(0, 0), (2, 6), \text{and} (-2, -6)$.

(b)\[
\begin{bmatrix}
1 & -3x_1^2 & 1 \\
3 & -1 \\
1 & 3 & -1
\end{bmatrix} \Rightarrow \lambda^2 - 4 = 0 \Rightarrow \lambda = \pm 2
\]

The equilibrium point $(0, 0)$ is unstable (saddle).

\[
\begin{bmatrix}
-11 & 1 \\
3 & -1
\end{bmatrix} \Rightarrow \lambda^2 + 12\lambda + 8 = 0 \Rightarrow \lambda = -11.29, \; -0.71
\]

The equilibrium point $(2, 6)$ is asymptotically stable (stable node).

\[
\begin{bmatrix}
-11 & 1 \\
3 & -1
\end{bmatrix}
\]
The equilibrium point \((-2, -6)\) is asymptotically stable (stable node).

(c) Let \(A = \begin{bmatrix} -11 & 1 \\ 3 & -1 \end{bmatrix}\) and \(P\) be the solution of \(PA + A^T P = -I\). Using Matlab, \(P\) is found to be
\[
P = \begin{bmatrix} 0.0938 & 0.1771 \\ 0.1771 & 0.6771 \end{bmatrix}.
\]
The eigenvalues of \(P\) are \(\lambda_{\max}(P) = 0.7266\) and \(\lambda_{\min}(P) = 0.0442\). To estimate the region of attraction of \((2,6)\), shift the equilibrium point to the origin via the change of variables
\[
\bar{x}_1 = x_1 - 2, \quad \bar{x}_2 = x_2 - 6
\]

The state equation in the new coordinates is given by
\[
\begin{align*}
\dot{\bar{x}}_1 &= -11\bar{x}_1 + \bar{x}_2 - 6\bar{x}_1^2 - \bar{x}_1^3 \\
\dot{\bar{x}}_2 &= 3\bar{x}_1 - \bar{x}_2
\end{align*}
\]

We use \(V = \bar{x}^T P \bar{x}\) as a Lyapunov function candidate. The derivative \(\dot{V}\) is given by
\[
\dot{V} = -\bar{x}^T \bar{x} - 2(p_{11}\bar{x}_1 + p_{12}\bar{x}_2)(6 + \bar{x}_1)\bar{x}_1^2
\]
\[
\leq -\|\bar{x}\|_2^2 - 12(p_{11}\bar{x}_1 + p_{12}\bar{x}_2)\bar{x}_1^2 - 2p_{12}\bar{x}_1^3 \bar{x}_2
\]
\[
\leq -\|\bar{x}\|_2^2 + 12\sqrt{p_{11}^2 + p_{12}^2}\|\bar{x}\|_2^3 + p_{12}\|\bar{x}\|_2^4
\]
\[
\leq -(1 - 2.4r - 0.1771r^2)\|\bar{x}\|_2^2, \quad \text{for} \quad \|\bar{x}\|_2 \leq r
\]

Taking \(r = 0.4\), we see that \(\dot{V}(\bar{x})\) is negative in \(\{\|\bar{x}\|_2 \leq r\}\). Choosing \(c < \lambda_{\min}(P)r^2 = 0.00707\), ensures that \(\{V(\bar{x}) \leq c\} \subset \|\bar{x}\|_2 \leq r\) because \(\lambda_{\min}(P)\|\bar{x}\|_2^2 \leq V(\bar{x})\). Take \(c = 0.007\). Thus, the region of attraction is estimated by \(\{\bar{x}^T P \bar{x} \leq 0.007\}\). The estimate of the region of attraction of \((-2, -6)\) is done similarly and the constant \(c\) is chosen to be 0.007. A less conservative estimate of the region of attraction can be obtained graphically by plotting the contour of \(\dot{V}(\bar{x}) = 0\) in the \(x_1-x_2\) plane and then choosing \(c\) and plotting the surface \(V(\bar{x}) = c\), with increasing \(c\), until we obtain the largest \(c\) for which the surface \(V(\bar{x}) = c\) is inside the region \(\{\dot{V}(\bar{x}) < 0\}\). The constant \(c\) is determined to be 0.1. The two estimates of the region of attraction are shown in Figure 4.2.

(d) The phase portrait is shown in Figure 4.3 together with the estimates of the region of attraction obtained in part (c). The stable trajectories of the saddle form a separatrix that divides the plane into two halves, with the right half as the region of attraction of \((2,6)\) and the left half the region of attraction of \((-2, -6)\). Notice that the estimates of the regions of attraction are much smaller that the regions themselves.

• 4.30 (a) The equilibrium points are the roots of the equations
Figure 4.4: Exercise 4.30. The dotted line is the contour of \( \dot{V}(y) = 0 \) and the solid line is the contour of \( V(y) = 0.07 \).

\[
\frac{\partial f}{\partial x}_{x=0} = \begin{bmatrix}
-2x_1 & x_2 & 1 - x_1^2 \\
-1 + 3x_1^2 + 2x_1x_2 & -(1 - x_1^2) & 0
\end{bmatrix}_{x=0} = \begin{bmatrix}
0 & 1 \\
-1 & -1
\end{bmatrix}, \quad \lambda_{1,2} = (-1 \pm j\sqrt{3})/2
\]

\[
\frac{\partial f}{\partial x}_{x=0} = \begin{bmatrix}
-1 & -1 \\
2 & -3x_2^2
\end{bmatrix}_{x=0} = \begin{bmatrix}
-1 & -1 \\
2 & 0
\end{bmatrix}, \quad \lambda_{1,2} = (-1 \pm j\sqrt{7})/2
\]

• 4.32 We investigate stability of the origin using linearization.

(1) \[
\frac{\partial f}{\partial x} = \begin{bmatrix}
-1 + 2x_1 & 0 & 0 \\
0 & -1 & 2x_3 \\
-2x_1 & 0 & 1
\end{bmatrix}, \quad A = \frac{\partial f}{\partial x}_{x=0} = \begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

A has an eigenvalue at 1; hence, the origin is unstable.

(2) Near the origin, \( sat(y) = y \) which implies that

\[
z \overset{\text{def}}{=} -2x_3 - sat(y) = 2x_1 + 5x_2 - 4x_3
\]

\[
\frac{\partial f}{\partial x} = \begin{bmatrix}
0 & 1 & 0 \\
2 & 10x_1 & -8x_1 - \cos x_3 \\
5 & -4 & 0
\end{bmatrix}, \quad A = \frac{\partial f}{\partial x}_{x=0} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & -1 \\
2 & 5 & -4
\end{bmatrix}
\]

The eigenvalues of \( A \) are \(-1, -1, -2\); hence, the origin is asymptotically stable.

(3) \[
\frac{\partial f}{\partial x} = \begin{bmatrix}
-2 + 3x_1^2 & 0 & 0 \\
2x_1 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}, \quad A = \frac{\partial f}{\partial x}_{x=0} = \begin{bmatrix}
-2 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

The eigenvalues of \( A \) are \(-2, -1, -1\); hence, the origin is asymptotically stable.

(4) \[
\frac{\partial f}{\partial x} = \begin{bmatrix}
-1 & 0 & 0 \\
1 - x_3 & -1 & 0 \\
x_2 & -1 - x_1 & 0
\end{bmatrix}, \quad A = \frac{\partial f}{\partial x}_{x=0} = \begin{bmatrix}
-1 & 0 & 0 \\
-1 & -1 & -1 \\
0 & 1 & 0
\end{bmatrix}
\]

The eigenvalues of \( A \) are \(-1, -\frac{1}{2} \pm j\frac{1}{2}\sqrt{3}\); hence, the origin is asymptotically stable.
4.44 Let $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$.

$$
V = -x_1^2 + x_1x_2 + x_1(x_1^3 + x_2^3)\sin t - x_1x_2 - x_2^2 + x_2(x_1^3 + x_2^3)\cos t
= -(x_1^2 + x_2^2) + (x_1^3 + x_2^3)(x_1\sin t + x_2\cos t)
\leq -||x||_2^2 + ||x||_2^3\sqrt{(\sin t)^2 + (\cos t)^2} = -||x||_2^2 + ||x||_2^3
\leq -(1-r)||x||_2^2, \quad \forall ||x||_2 \leq r, \text{ for any } r < 1
$$

Hence, by Theorem 4.10, the origin is exponentially stable. Since $V(x) = \frac{1}{2}||x||_2^2$, the region of attraction can be estimated by the set \{||x||_2 \leq r\} for any $r < 1$.

4.45 (a) Use $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ as a Lyapunov function candidate.

$$
\dot{V} = x_1[h(t)x_2 - g(t)x_1^3] + x_2[-h(t)x_1 - g(t)x_2^3] = -g(t)(x_1^4 + x_2^4) \leq -k(x_1^4 + x_2^4)
$$

Hence, the origin is uniformly asymptotically stable. Since all assumptions hold globally and $V(x)$ is radially unbounded, the origin is globally uniformly asymptotically stable, which answers part (c).

(b) The conditions of Theorem 4.10 are not satisfied. Let us linearize the system

$$
A(t) = \frac{\partial f}{\partial x}(t,0) = \begin{bmatrix}
0 & h(t) \\
-h(t) & 0
\end{bmatrix}
$$

Use $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ as a Lyapunov function candidate for the linear system.

$$
\dot{V} = x_1h(t)x_2 - x_2h(t)x_1 = 0
$$

This shows that solutions starting on the surface $V(x) = c$ remain on that surface for all $t$. Hence, the origin of the linear system is not exponentially stable. This implies that the origin of the nonlinear system is not exponentially stable.

(d) No.

4.46 Linearization at the origin yields the matrix

$$
\left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix}
-1 + 3x_1^2 + x_2^2 & -1 + 2x_1x_2 \\
1 + 2x_1x_2 & -1 + x_1^2 + 3x_2^2
\end{bmatrix}
= \begin{bmatrix}
-1 & -1 \\
1 & -1
\end{bmatrix}
$$

whose eigenvalues are $-1 \pm j$. Hence, the matrix is Hurwitz and the origin is exponentially stable. Consequently, it is asymptotically stable.

4.47 Let $V(x) = \frac{1}{2}(bx_1^2 + ax_2^2)$.

$$
\dot{V} = -b\phi(t)(x_1 - ax_2)^2 - ac\psi(t)x_2^4 \leq -b\phi_0(x_1 - ax_2)^2 - ac\psi_0x_2^4 \equiv -W_3(x)
$$

It can be verified that $W_3(x)$ is positive definite for all $x$. Hence, by Theorem 4.9, the origin is globally uniformly asymptotically stable. Linearization at the origin yields the linear system

$$
\dot{y}_1 = -\phi(t)y_1 + a\phi(t)y_2, \quad \dot{y}_2 = b\phi(t)y_1 - ab\phi(t)y_2
$$

The invertible change of variables

$$
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
b & 1 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
y_1 \\
y_2
\end{bmatrix}
$$

results in the system

$$
\dot{z}_1 = 0, \quad \dot{z}_2 = \phi(t)z_1 - \phi(t)(1 + ab)z_2
$$

which has the solution $z_1(t) \equiv \text{constant}$. This shows that the linear system is not exponentially stable. Therefore, the origin of the nonlinear system is not exponentially stable.