• 4.1

(1) asymptotically stable  (2) unstable  (3) asymptotically stable  (4) unstable  (5) stable  (6) unstable

• 4.2 Let \( f(x) = ax^p + g(x) \). Near the origin, the term \( ax^p \) is dominant. Hence, \( \text{sign}(f(x)) = \text{sign}(ax^p) \). Consider the case when \( a < 0 \) and \( p \) is odd. With \( V(x) = \frac{1}{2}x^2 \) as a Lyapunov function candidate, we have

\[
\dot{V} = x[ax^p + g(x)] \leq ax^{p+1} + k|x|^{p+2}
\]

Near the origin, the term \( ax^{p+1} \) is dominant. Hence, \( \dot{V}(x) \) is negative definite and the origin is asymptotically stable. Consider now the case when \( a > 0 \) and \( p \) is odd. In the neighborhood of the origin, \( \text{sign}(f(x)) = \text{sign}(x) \). Hence, a trajectory starting near \( x = 0 \) will be always moving away from \( x = 0 \). This shows that the origin is unstable. When \( p \) is even, a similar behavior will take place on one side of the origin; namely, on the side \( x > 0 \) when \( a > 0 \) and \( x < 0 \) when \( a < 0 \). Therefore, the origin is unstable.

• 4.3 (1) Let \( V(x) = (1/2)(x_1^2 + x_2^2) \).

\[
\dot{V} = x_1(-x_1 + x_1x_2) - x_2^2
\]

In the set \( \{ ||x||_2 \leq r^2 \} \), we have \( |x_1| \leq r \). Hence,

\[
\dot{V} \leq -x_1^2 - x_2^2 + r|x_1| |x_2| = -\begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}^T \begin{bmatrix} 1 & -r/2 \\ -r/2 & 1 \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}
\]

\( \dot{V} \) is negative definite for \( r < 2 \). Thus, the origin is asymptotically stable. To investigate global asymptotic stability, note that the solution of the second equation is \( x_2(t) = \exp(-t)x_2(0) \), which when substituted in the first equation yields

\[
\dot{x}_1 = [-1 + \exp(-t)x_2(0)]x_1
\]

This is a linear time-varying system whose solution does not have a finite escape time. After some finite time the coefficient of \( x_1 \) on the right-hand side will be less than a negative number. Hence, \( \lim_{t \to \infty} x_1(t) = 0 \). Thus, the origin is globally asymptotically stable.

(2) Let \( V(x) = (1/2)(x_1^2 + x_2^2) \).

\[
\dot{V} = -(x_1^2 + x_2^2)(1 - x_1^2 - x_2^2) = -2V(1 - 2V)
\]

In the region \( V(x) < 1/2 \), \( \dot{V} \) is negative definite. Hence, the origin is asymptotically stable. For \( V > 1/2 \), \( \dot{V} \) is positive. Hence, trajectories starting in the region \( V(x) > 1/2 \) cannot approach the origin. In fact, they grow unbounded. Thus, the origin is not globally asymptotically stable.

(3) Let \( V(x) = x^TPx = p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2 \), where \( P \) is a positive definite symmetric matrix.

\[
\dot{V} = -2p_{12}x_1^2 + 2(p_{11} - p_{12} - p_{22})x_1x_2 - 2(p_{22} - p_{12})x_2^2 + \text{Higher order terms}
\]
Near the origin, the quadratic term dominates the higher-order terms. Thus, \( \dot{V} \) will be negative definite in the neighborhood of the origin if the quadratic term is negative definite. Choosing \( p_{12} = 1, p_{22} = 2, \) and \( p_{11} = 3 \) makes \( V(x) \) positive definite and \( \dot{V}(x) \) negative definite. Hence, the origin is asymptotically stable. It is not globally asymptotically stable since the origin is not the unique equilibrium point. The set \( \{x_1^2 = 1\} \) is an equilibrium set.

(4) Let \( V(x) = x_1^2 + (1/2)x_2^2. \)

\[
\dot{V} = -2x_1^2 - 2x_1x_2 + 2x_1x_2 - x_2^4 = -x_1^2 - x_2^4
\]

Hence, the origin is globally asymptotically stable.

• 4.4 (a) Take \( V(\omega) = (1/2)(J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2) \) as a Lyapunov function candidate.

\[
\dot{V} = J_1\omega_1\dot{\omega}_1 + J_2\omega_2\dot{\omega}_2 + J_3\omega_3\dot{\omega}_3 \\
= (J_2 - J_3)\omega_1\omega_2\omega_3 + (J_3 - J_1)\omega_1\omega_2\omega_3 + (J_1 - J_2)\omega_1\omega_2\omega_3 \\
= 0
\]