Existence & Uniqueness (cont.)

Example: \( f(x) = \begin{bmatrix} x_1^2 \\ -\text{sat}(x_1 + x_2) \end{bmatrix} \) is cont. but not (cont.-)diff. on \( \mathbb{R}^2 \),
(but Lipschitz in \( \mathbb{R}^2 \))

\[
\|f(x) - f(y)\|_2 = \left\| \begin{bmatrix} x_1 - y_1 \\ \text{sat}(x_1 + y_2) - \text{sat}(x_1 + y_2) \end{bmatrix} \right\|_2 \\
\leq (x_1 - y_1)^2 + (\text{sat}(x_1 + y_2) - \text{sat}(x_1 + y_2))^2 \\
\leq (x_2 - y_2)^2 + ((x_1 + x_2) - (y_1 + y_2))^2 \\
\leq 2(x_1 - y_1)^2 + 2(x_1 - y_1)(x_2 - y_2) + 2(x_2 - y_2)^2 \\
\leq 2(x_1 - y_1)^2 + 3(x_2 - y_2)^2 \\
\leq 3\|x - y\|_2^2
\]

\( \Rightarrow \|f(x) - f(y)\|_2 \leq \sqrt{3}\|x - y\|_2 \). Thus globally Lipschitz (weaker than cont. diff.).

Locally Lipschitz at \((t_0, x(t_0))\) guarantees unique solution in nbhd of \((t_0, x(t_0))\),
but may extend up to \(t = t_0 + \delta\).
Further extension would require local Lipschitzness at \((t_1, x(t_1))\), etc. In general exist a max. \( T \) s.t. unique solution exists over \([t, T] \).
As \( t \to T \), the solution leaves any compact set over which \( f \) is locally Lipschitz.

Example: \( x = -x^2 \), \( x(0) = -1 \)
Here \( f = x^2 \) is cont. & cont. diff. but differential not uniformly held.
However diff. bdd on any compact set \( \Rightarrow \) Lipschitz over that compact set.
Unique solution \( x(t) = \frac{1}{t+1} \) exists over \([0, 1)\). As \( t \to 1 \), \( x \) leaves any compact set.

Question: When can the unique solution exist indefinitely?

Thm 2: \( \dot{x} = f(t, x) \) has unique solution over \([t_0, t] \) if \( f \) Lipschitz over \([t_0, t] \times \mathbb{R}^n \) and piecewise-cont. in \( t \) over \([t_0, t] \).

"Lipschitz" requirement of above thm is restrictive: \( \dot{x} = -x^3 = -f(x) \).
Here \( f \) is cont. & cont. diff., but \( \frac{df}{dt} \) not bounded. Yet unique solution exists:
\[
x(t) = \exp(x(t_0)) \sqrt{t^2 + \frac{2x^2(t_0)}{t-t_0}} + t > t_0.
\]
Example: \( \dot{x} = A(x) x + g(t) \)

\[
\Rightarrow \| f(t, x) - f(t, y) \| = \| A(x)(x-y) \| \leq \| A(x) \| \| x - y \|
\]

So if \( A(x) \) is bounded for \( t \in [t_0, t_1] \), we have that conditions of Thm 2 hold.

As we discussed, condition of Thm 2 are quite strong, and so here is another result:

**Thm 3:** \( \dot{x} = f(t, x) \) has unique solution for all \( t \geq t_0 \) if

- \( f \) locally Lipschitz over \( [t_0, x_0] \times U \), \( U \subseteq \mathbb{R}^n \) compact set,
- \( x_0 \in U \), and solution of \( \dot{x} = f(t, x) \) does not exit \( U \).

Example: \( \dot{x} = -x^3 = f(x) \). Then \( f \) is locally Lipschitz over \( [t_0, x_0] \times \mathbb{R}^n \).

Also if \( x_0 = 0 \), then system never leaves the set \( \{ x \in \mathbb{R}^n : |x| \leq 1 \} \).

(This is because \( x > 0 \Rightarrow \dot{x} < 0 \), and \( \dot{x} < 0 \Rightarrow \dot{x} > 0 \).)

**Continuous dependence on initial conditions/parameters**

- Does small change in \( t_0, x_0 \), or \( f \) causes small change in solution? 
- Continuous dependence on \( t_0 \) since,

\[
x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s)) \, ds
\]

- \( \text{cont. dependence on } x_0 : \) Suppose \( \dot{x} = f(t, x) \) uniquely solvable over \( [t_0, t_1] \). Starting \( x_0 \).

  \( \forall t \in \mathbb{R}^n - x_0 \), \( \| x(t) - x_0 \| \leq \varepsilon \) \( \forall t \in [t_0, t_1] \) and \( x(t) \) unique.

- \( \text{cont. dependence on } f : \) \( \{ f_n \} \xrightarrow{m=\infty} f \) uniformly in \( t \Rightarrow \{ x_m \} \xrightarrow{m=\infty} x \) uniformly?

Another way to study cont. dependence on \( f \), parametrize \( f \) using parameter \( \lambda \).

\( \Rightarrow \dot{x} = f(t, x, \lambda) \), which suppose has solution over \( [t_0, t_1] \).

\( \forall \lambda, \| x_0 - \lambda \| \leq \theta \Rightarrow \| x(t, \lambda) - x(t, \lambda_0) \| \leq \varepsilon \) \( \forall t \in [t_0, t_1] \) and \( x(t, \lambda) \) open, connected.

**Thm:** \( f(t, x, \lambda) \) cont. in \( (t, x, \lambda) \) & locally Lipschitz over \( [t_0, t_1] \times D \{ \| x - \lambda \| \leq \theta \} \)

\( \forall \varepsilon \in \mathbb{R} : \) \( |x_0 - \lambda_0| < \delta \Rightarrow \| x(t, \lambda) - x(t, \lambda_0) \| < \varepsilon \) \( \forall t \in [t_0, t_1] \).

\( \varepsilon \)-tube around \( x(t_0, \lambda) \)
Differentiability of Solution & Sensitivity Eq.

- Under the additional requirement that \( f(t,x,\lambda) \) is continuously differentiable in \( x,\lambda \) (instead of just satisfying some Lipschitz condition) over \([t_0,t_1] \times \mathbb{R}^n \times \mathbb{R}^m\), then \( z(t,\lambda) \) is differentiable with \( \frac{\partial z}{\partial \lambda} \) near \( x_0, \lambda_0 \) where \( x_0, \lambda_0 \) such that \( \dot{z}=f(t,x,\lambda_0) \) with \( z(t_0)=x_0 \) has unique soln. over \([t_0,t_1]\).

- Further \( \dot{z}(t,\lambda) = \dot{z}(t,\lambda_0) + \dot{z}(t,\lambda_0) \), where
  \[
  \dot{z}(t,\lambda) = \left( \frac{\partial f}{\partial x} \right)_{t=t_0, \lambda=\lambda_0} (z(t,\lambda) - z(t,\lambda_0)) + \left( \frac{\partial f}{\partial \lambda} \right)_{t=t_0, \lambda=\lambda_0} \lambda - \lambda_0
  \]

- Thus if \( z(t,\lambda_0) \) is available as solution of \( \dot{z}=f(t,x,\lambda_0), z(t_0)=x_0 \), then \( z(t,\lambda) \) can be obtained by first solving eq. for "sensitivity".

- Another way to approach this is by solving the following together:
  \[
  \dot{z}=f(t,x,\lambda_0) \quad \text{with} \quad z(t_0)=x_0
  \]
  \[
  \dot{\lambda} = \left( \frac{\partial f}{\partial \lambda} \right)_{t=t_0, \lambda=\lambda_0} \lambda - \lambda_0
  \]

- These are usually solved numerically.

Example: Phase-locked-loop
  \[
  \dot{x_1} = x_2 \\
  \dot{x_2} = -c \sin x_1 - (a+b \cos x_1) x_2
  \]

\( \lambda = \begin{bmatrix} a \\ b \end{bmatrix} \) with \( \lambda_0 = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} \)

\( \Rightarrow f(x,\lambda_0) = \begin{bmatrix} x_2 \\ -c \sin x_1 - x_2 \end{bmatrix} \)

Also, \( \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -c \cos x_1 + b_0 \sin x_1 & -(a+b \cos x_1) \end{bmatrix} \) and \( \frac{\partial f}{\partial \lambda} = \begin{bmatrix} -x_2 & 2 \lambda_0 \sin x_1 & 0 \end{bmatrix} \)

\( \Rightarrow \frac{\partial f}{\partial \lambda} = \begin{bmatrix} -x_2 \end{bmatrix} \) (at \( \lambda = \lambda_0 \))

\( \dot{\lambda} = \begin{bmatrix} 0 & 1 \\ -c \cos x_1 & -1 \end{bmatrix} \)

\( \dot{\lambda} = \begin{bmatrix} 0 & 1 \\ -c \cos x_1 & -1 \end{bmatrix} \dot{\lambda} + \begin{bmatrix} 0 & 0 & 0 \\ -a_0 \sin x_1 & -b_0 \sin x_1 & -\sin x_1 \end{bmatrix} \) (s = \( s_{2 \times 3} \))
More on differentiability of solution & Sensitivity Equation

\[ \dot{z} = f(t, z, \lambda) \quad \text{with} \quad z(t_0) = z_0 \]

\[ \Rightarrow z(t, \lambda) = z_0 + \int_{t_0}^{t} f(s, z(s, \lambda), \lambda) \, ds \]

\[ \Rightarrow \frac{\partial}{\partial \lambda} z(t, \lambda) = \int_{t_0}^{t} \left[ \frac{\partial f}{\partial z} (s, z(s, \lambda), \lambda) z_\lambda(s, \lambda) + \frac{\partial f}{\partial \lambda} (s, z(s, \lambda), \lambda) \right] \, ds \]

Assumes \( f \) is diff. w.r.t. \( z, \lambda \)

Since \( z_\lambda(t, \lambda) \) is given as an integral \( \Rightarrow \frac{\partial}{\partial t} z_\lambda(t, \lambda) \) differentiable w.r.t. \( t \), if \( f \) is a cont. function (assumes \( f \) is cont. diff. w.r.t. \( z, \lambda \))

\[ \Rightarrow \frac{\partial}{\partial t} z_\lambda(t, \lambda) = \frac{\partial f}{\partial z} (t, z(t, \lambda), \lambda) z_\lambda(t, \lambda) + \frac{\partial f}{\partial \lambda} (t, z(t, \lambda), \lambda) \]

\[ \Rightarrow \dot{s} = A(t, \lambda) \, s + B(t, \lambda) \]

\[ s(t_0) = \int_{t_0}^{t} \cdots \, ds \Rightarrow s(t_0) = 0 \]

\( s(t) \): Sensitivity function