2nd Order Systems

- 2nd order systems significant since trajectories can be plotted on 2-D plane = visual examination possible
  - $x_1, x_2$ plane: Phase plane or State plane.
  - 2nd order $\Rightarrow \begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}$ assuming TI & autonomous
  \[ x = \frac{\dot{f}_2}{f_1} \]

- Trajectory starting at $x_0$: Locus of $\dot{x}(t)$ starting at $x(0) = x_0$

- Slope of trajectory in phase plane $= \frac{dx_2}{dx_1} = \frac{d\dot{x}_2}{d\dot{x}_1} = \frac{f_2}{f_1}$

  - The vector $f = [f_1, f_2]$ has the same slope $= \tan \theta = \frac{f_2}{f_1}$.

- Thus by plotting vector $f$ at several points in $x_1, x_2$ plane we can approximately plot trajectory. Such a plot called "vector field".

- Pendulum's vector field:
  \[ \begin{align*}
  \dot{x}_1 &= x_2 \\
  \dot{x}_2 &= -10 \sin x_1
  \end{align*} \]

  Length of arrow proportion to $|f| = \sqrt{f_1^2 + f_2^2}$ at each point.

- Family of all trajectories called phase portrait, which can be obtained by drawing trajectories for several initial states.

- Note: a phase portrait shows $x_1, x_2$ plot, and does not show the "motion" as $t$ evolves. So it is "qualitative" in nature. The quantitative information (motion as fn. of time) is not included.
Phase Portrait of 2nd order Linear Systems

- 2nd order linear \( \dot{x} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} x = Ax = (M J_r M^{-1})x \)

Equilibrium: \( Ax = 0 \) \( \Rightarrow x = 0 \) if \( \det(A) \neq 0 \), else eq. a subspace.

\( J_r \): Jordan form can be of these three forms.

\[
\begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
\lambda & 1 \\
0 & \lambda
\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
\alpha & -\beta \\
\beta & \alpha
\end{bmatrix}
\]

distinct real eigen values 
repeated real eigen values \((k = 0 \text{ or } 1)\) 
complex eigen values \(\lambda_{1,2} = \alpha \pm j\beta\).

\( M \): Matrix of (extended) eigen vectors of \( A \).

\( \dot{x} = MJ_r M^{-1}x \Rightarrow M^{-1}\dot{x} = J_r M^{-1}x \Rightarrow \dot{\tilde{x}} = J_r \tilde{x} \) \( (\tilde{z} = M^{-1}z \text{ model coordinates})\).

**CASE 1:** \( \lambda_1 \neq \lambda_2 \neq 0 \) \( \Rightarrow M = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \) with \( Mv_1 = \lambda_1 v_1 \).

Also \( \dot{z}_1 = \lambda_1 z_1 \Rightarrow z_1(t) = e^{\lambda_1 t} z_1(0) \Rightarrow z_2 = c \frac{z_2(0)}{z_1(0)} e^{\lambda_2 t} \)

1.1 \( \lambda_1, \lambda_2 < 0 \); WLOG \( \lambda_2 < \lambda_1 < 0 \) (\( \lambda_2 \): "faster", \( \lambda_1 \): "slower")

\[
\frac{d^2 z_2}{dz_1^2} = c \frac{\lambda_2}{\lambda_1} \frac{z_2}{z_1} (\frac{\lambda_2}{\lambda_1} - 1)
\]

\( z_2(t) \to 0 \) as \( t \to \infty \) \( \Rightarrow \) origin stable

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=> trajectory tangential to \( z_1 \) near origin, perpendicular to \( z_2 \)

(or parallel to \( z_2 \)) away from origin

1.2 \( \lambda_1, \lambda_2 > 0 \):

WLOG \( \lambda_2 > \lambda_1 > 0 \)

\( z_2(t) \to \infty \) as \( t \to \infty \) \( \Rightarrow \) origin unstable

\( z_1(t) \to 0 \) as \( t \to \infty \) \( \Rightarrow \) phase portrait same character but origin has unstable trajectories reversed.
Phase portrait of 2nd order linear systems

1.3 \( A_1, A_2 \) opposite sign: \( \text{WLOG } A_2 < 0 < A_1 \).

\[
\frac{dz}{dt} = \begin{pmatrix} A_1 & -A_2 \\ A_2 & 0 \end{pmatrix} z
\]

- exponent of \( z_1 \) is -ve
- slope \( \to \infty \) \( |z_1| \to \infty \)
- slope \( \to \infty \) \( |z_2| \to \infty \)

\( \to \) near origin parallel to \( z_2 \), away from origin tangent to \( z_1 \).

![Phase plane](image)

![Modal plane](image)

Hyperbolic shape, except along \( z_1, z_2 \): unstable \( \frac{z_2}{\lambda_2} \): stable

\( \text{Origin in the above case is a "saddle point".} \)

\text{Case 2 } \lambda_1, \lambda_2 = \alpha \pm j\beta

\[
\begin{align*}
\dot{z}_1 &= \alpha z_1 + \beta z_2 \\
\dot{z}_2 &= \beta z_1 + \alpha z_2
\end{align*}
\]

- consider \( (r, \theta) = \left( \sqrt{z_1^2 + z_2^2}, \tan^{-1}\left( \frac{z_2}{z_1} \right) \right) \), i.e., polar coordinates

\( \to \)

\[
\begin{align*}
r^2 &= z_1^2 + z_2^2 \\
r \dot{r} &= z_1 \dot{z}_1 + z_2 \dot{z}_2
\end{align*}
\]

\[
\begin{align*}
r &= \frac{z_1 (\alpha z_1 - \beta z_2) + z_2 (\beta z_1 + \alpha z_2)}{z_1^2 + z_2^2} \\
\dot{r} &= \alpha r^2 - \beta z_1 \dot{z}_2 + \beta z_2 \dot{z}_1 + \alpha z_2^2 = \alpha (z_1^2 + z_2^2)
\end{align*}
\]

\( \Rightarrow \begin{pmatrix} r \\ \theta \end{pmatrix} = e^{\alpha t} \begin{pmatrix} r(0) \\ \theta(0) \end{pmatrix} \)

Also, \( \tan \theta = \frac{z_2}{z_1} \)

\[
\begin{align*}
\dot{z}_1 \sin \theta + \dot{z}_2 \cos \theta &= 0 \\
\dot{z}_2 \sin \theta + z_1 \cos \theta \dot{\theta} &= \dot{z}_2 \cos \theta - z_2 \sin \theta \dot{\theta} \\
\Rightarrow & \ (z_1 - \beta z_2) \frac{z_2}{z_1} + z_2 \dot{\theta} = (\beta z_1 + \alpha z_2) - \frac{z_2}{z_1} \dot{\theta}
\end{align*}
\]

\[
\Rightarrow \frac{z_1^2 + z_2^2}{z_2} \dot{\theta} = \frac{\beta^2 + z_2^2}{z_1} \Rightarrow \dot{\theta} = \frac{\beta}{z_2} \theta \Rightarrow \theta(t) = \beta t + \theta(0)
\]
Phase portrait of 2nd order linear system

\( \dot{x} = e^{\alpha t} x(0), \quad \beta t + \theta(0) \rightarrow \) exponential spiral.
\( \alpha = 0 \rightarrow \) radial size constant (origin a "center")
\( \alpha > 0 \rightarrow \) radius increases exponentially with time (origin a stable focus)
\( \alpha < 0 \rightarrow \) radius decreases exponentially with time (origin an unstable focus)
\( \beta > 0 \rightarrow \) angle rotates counterclockwise linearly with time
\( \beta < 0 \rightarrow \) angle rotates clockwise linearly with time.

**Case III** \( \lambda_1 = \lambda_2 = \lambda \neq 0 \)
\( \dot{z}_1 = \lambda z_1 + k \dot{z}_2 \quad \dot{z}_2 = \lambda z_2 \)

\( \Rightarrow z_2(t) = e^{\lambda t} z_2(0), \quad z(t) = e^{\lambda t} z(0) + \int e^{\lambda t} k \vec{z}(0) z_1(0) \, dz \)
\( \Rightarrow t = \frac{1}{\lambda} \ln \frac{z_2(t)}{z_2(0)} \)
\( \Rightarrow z(t) = e^{\lambda t} z(0) + e^{\lambda t} \int_0^t k \vec{z}(0) \vec{z}(0) \, dz \)
\( = e^{\lambda t} z(0) + e^{\lambda t} \int_0^t k \vec{z}(0) \vec{z}(0) \, dz \)
\( = e^{\lambda t} [z(0) + k \vec{z}(0) t]. \)

Also, \( z(t) = \frac{z_2(t)}{z_2(0)} = \frac{z_2(t)}{z_2(0)} \left[ z_1(0) + k \frac{z_2(0)}{z_2(0)} \ln \left( \frac{z_2(t)}{z_2(0)} \right) \right] \)
\( = z_1 \left[ \frac{z_2(t)}{z_2(0)} + k \frac{z_1(0)}{z_2(0)} \ln \left( \frac{z_2(t)}{z_2(0)} \right) \right]. \)
Phase portrait of 2nd order linear system

III.1 \( k = 0 \Rightarrow x_1 = x_2 \left( \frac{z_1(0)}{z_2(0)} \right) \)
\[ \lambda < 0 \Rightarrow x_1, x_2 \to 0 \text{ as } t \to \infty \]
\[ \lambda > 0 \Rightarrow x_1, x_2 \to \infty \text{ as } t \to \infty \]

III.2 \( k = 1 \Rightarrow x_1 = x_2 \left( \frac{z_1(0)}{z_2(0)} + \frac{1}{2} \ln \left( \frac{z_2(0)}{z_1(0)} \right) \right) \lambda > 0 \)
\[ \lambda < 0 \Rightarrow x_1, x_2 \to 0 \text{ as } t \to \infty \]
\[ \lambda > 0 \Rightarrow x_1, x_2 \to \infty \text{ as } t \to \infty \]

Origin stable node when \( \lambda < 0 \), unstable node when \( \lambda > 0 \).

CASE IV one or both eigenvalue zero
Equilibrium set is subspace with dimension 1 (one eigenvalue zero)
Equilibrium set is subspace with dimension 2 (both eigenvalues zero)
\[ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \text{no motion} \ (z_1(t) = z_1(0), z_2(t) = z_2(0)) \]
\[ \lambda_1 = 0, \lambda_2 \neq 0 \Rightarrow \dot{x}_1 = 0, \dot{x}_2 = \lambda_2 t \Rightarrow \text{solution} \ (z_1(t) = z_1(0), z_2(t) = e^{\lambda_2 t} z_2(0)) \]
\[ \lambda_1, \lambda_2 \neq 0 \Rightarrow \text{not null space vector} \]
\[ \lambda_1, \lambda_2 = 0 \Rightarrow \text{null space vector} \]

\[ \lambda_2 < 0 \]
\[ \lambda_2 \neq 0 \Rightarrow \dot{z}_2(t) \to 0 \text{ as } t \to \infty \]
\[ \text{Since } \dot{z}_1 \text{ does not change, vertical motion in mode plane (motion parallel to } v_2 \text{ in phase plane)} \]
\[ \lambda_2 > 0 \Rightarrow \dot{z}_2(t) \to \infty \text{ as } t \to \infty \]
\[ \lambda_1 = \lambda_2 = 0 \Rightarrow \dot{z}_1 = 0, \dot{z}_2 = 0 \Rightarrow \dot{z}_2(t) = z_2(0) \text{ and } z(t) = z_2(0) t. \]