Invariance Principle

In pendulum example, \( \ddot{\theta} = \ddot{x} \),
\[ \ddot{x} = -a \sin x - bx, \quad V(x) = \frac{1}{2} \dot{x}^2 + a(1 - \cos x) \]

is not adequate to show asymptotic convergence since \( \dot{V}(x) = -\frac{1}{2} bx^2 \leq 0 \),
which implies \( \dot{V}(x) = 0 \) whenever \( x = 0 \) (\( V \) is not strictly convex).

However, \( \dot{V}(x) = 0 \Rightarrow \dot{x} = 0 \Rightarrow x = 0 \Rightarrow \sin x = 0 \Rightarrow x \approx 0 \) (assuming \( a > 0 \)).
So \( \dot{x} = 0 \), \( V(x) \) must decrease. (This is expected in presence of friction.)

If \( x \) exists \( V(x) \) with \( V(x) \leq 0 \) around origin, and \( \dot{V}(x) = 0 \Rightarrow x = 0 \), then origin must be asymptotically stable (known as LaSalle's Invariance Principle).

**Definitions:**
- \( \lim_{n \to \infty} \) the limit pt. of \( x(t) \) if \( \exists \{ t_n \} \) s.t. \( \{ x(t_n) \} \to p \)
- Set of all \( n \) limit pts is called the limit set.
- \( M \) is nonempty if \( x(t) \in M \Rightarrow x(t) \in M \) for all \( t \geq 0 \).
- \( M \) is nonempty if \( x(t) \to M \) if \( \| x(t) - x \| < \epsilon \) for all \( t \geq T \).

**Example:**
- Stable eq. pt. the limit pt. of points near the eq. pt.
- Stable limit cycle the limit set of points near the limit cycle.
- Also \( x(t) \) approaches stable eq. pt. or stable limit cycle.
- Eq. pt. or limit cycle are invariant sets.

**Lemma:** \( x(t) \) bounded and contained in \( D \) for \( t \geq 0 \), then \( x(t) \) has a +ve-limit set \( L^+ \) that is nonempty, compact, and invariant. Also \( x(t) \to L^+ \).

**Inv. Principle:** \( V: D \to R \) cont. diff., \( x \in C \) compact, s.t. \( \dot{V}(x) \leq 0 \) on \( L \).
\( E = \{ x \in L | V(x) = 0 \} \) and \( M \subseteq E \) largest inv. set. Then \( \forall \theta \in E \to x(t) \to M \).

Pick \( x(t) \in E \to x(t) \in L^+ \) and \( \dot{V}(x) \) decreases monotonically\( \Rightarrow \frac{d}{dt} V(x) \to a \)
- \( V(x) \) cont. on \( L \), \( L \) compact \( \Rightarrow \) \( V(x) \) is lower bounded.

Also from Lemma, \( 3 L^+ \) s.t. \( L^+ \) nonempty, compact, iniv. Further \( L^+ \subset J \)
- Since \( L \) is closed (and \( L \) contains all limit points),
- \( \forall \theta \in L^+ \exists \{ x(t_n) \} \in L^+ \) \( \Rightarrow \frac{V(x(t_n))}{n} \to V(p) = a \)
- \( \Rightarrow V(p) = a \) for all \( \theta \in L^+ \). \( L^+ \) is invariant \( \Rightarrow x(t) \in L^+ \to x(t) \in L^+ \), \( \dot{V}(x) = 0 \) for \( x(t) \in L^+ \)
- \( L^+ \subseteq E \). \( M \) largest inv. subset of \( E = \bigcap L^+ \subset M \subset E \).
- \( a \to L^+ \to M \).
Invariance Principle (Ctdn.)

- In the Inv. theorem, $V(\mathbf{x}) > 0$ not required.
- Also $\mathcal{L}$ is not necessarily based on $V$. In many applications, $V$ itself provides $\mathcal{L}$, e.g., $\Omega_0 = \{ \mathbf{x} \mid V(\mathbf{x}) \leq c \}$ may be bounded and $V(\mathbf{x}) \leq 0$ over $\Omega_0$.

Then choose $\mathcal{L} = -\nabla V$.
- Also $V > 0 \Rightarrow \exists \ c > 0 \ s.t. \ \Omega_0$ is bounded (not necessarily true always) for radially unbounded $V$, $\Omega_0$ is bounded for all $c$.

**Corollary:** $V: \mathbb{R} \to \mathbb{R}$ cont. diff., $\mathcal{L}$-definite over $\mathbb{R} \ni 0$ s.t. $V(\mathbf{x}) \leq 0$ in $\Omega$. $E = \{ \mathbf{x} \in \mathbb{R} \mid V(\mathbf{x}) = 0 \}$ is such that no solution can stay in $E$ except $\mathbf{x}(t) = 0$. Then origin is asymptotically stable (M (largest inv. subset of $E$) = $\{ 0 \}$).

**Corollary:** $V: \mathbb{R}^n \to \mathbb{R}$ cont. diff., radially unbounded, $\mathcal{L}$-definite s.t. $V(\mathbf{x}) \leq 0$. $E = \{ \mathbf{x} \in \mathbb{R}^n \mid V(\mathbf{x}) = 0 \}$ is such that no solution can stay in $E$ except $\mathbf{x}(t) = 0$. Then origin globally asymptotically stable. (M (largest inv. subset of $E$) = $\{ 0 \}$)

**Example (generalized pendulum):**

\[
\begin{align*}
\dot{x}_1 &= \dot{x}_2 \\
\dot{x}_2 &= -h(x_2) - h_2(x_2) \\
V(x) &= \int_0^{\tau_2} h_2(t) \, dt + \frac{1}{2} \tau_2^2
\end{align*}
\]

$V(\mathbf{x}) = h(x_1) x_2 + x_2 [\tau_2 h(x_2) - h(\tau_2)] < -x_2 h(\tau_2) \leq 0$

$\dot{x}_2 = 0 \Rightarrow x_2 h(\tau_2) = 0 \Rightarrow x_2 = 0$ (since $h(x_2) > 0$ and $h(\tau_2) > 0$, $x_2 \in (-a, a)$)

Thus $E = \{ \mathbf{x} \in \mathbb{R}^2 \mid V(\mathbf{x}) = 0 \} = \{ x_2 = 0 \}$. Also $x_2(0) = 0 \Rightarrow \tau_2(0) = 0 \Rightarrow h(x_2(t)) = 0 \Rightarrow x_2(t) \equiv 0$.

The only solution that can stay in $E$ is $0$. From Lasalle's thm, $0$ asymptotically stable.

**Lasalle's Thm:**

1. Relaxes the requirement that $V(\mathbf{x}) < 0$
2. Region of attraction can be approximated as $\mathcal{L}$, a set with form different from $\Omega_0 = \{ \mathbf{x} \mid V(\mathbf{x}) \leq c \}$.
3. Does not require existence of isolated eq. pt. (can be eq. set)
4. $V(\mathbf{x})$ need not be $> 0$. 

\[
\int_0^{\tau_2} h(t) \, dt \to \infty \Rightarrow V(\mathbf{x}) \text{radically unbounded}
\]

Also it can be shown that $\dot{V}(\mathbf{x}) \leq 0$ for $\mathbf{x} \in \mathbb{R}^2$, and $E = \{ \mathbf{x} \in \mathbb{R}^2 \mid V(\mathbf{x}) = 0 \} = \{ x_2 = 0 \}$ contains only the trivial solution $\mathbf{x}(t) \equiv 0$.

Thus, the origin is globally asymptotically stable.
Invariance Principle Example

Example: 1st order system \( \{ y = ay + u \) \)
\[
\begin{align*}
u &= -ky, \quad k = \gamma y^2, \quad \gamma > 0 \\
x_1 &= y, \quad x_2 = k \\ x_1 &= -\gamma x_1 + u \\ x_2 &= x_2
\end{align*}
\]

At equilibrium, \((a-\gamma x_1) x_1 = \gamma x_1^2 = 0 \Rightarrow \{ x_1 = 0 \}\) is equilibrium set.
To show that trajectory approaches the set \( x_1 = 0 \) (adaptive controller regulates output to zero), let \( V(x) = \frac{1}{2} x_1^2 + \frac{1}{2\gamma} (x_2 - b)^2, \quad b > a \).
\[
V(x) = x_1 x_1 + \frac{1}{2\gamma} (x_2 - b) x_2 = (a-\gamma x_1) x_1^2 + (x_2 - b) x_2^2 = -x_1^2 (b-a) \leq 0
\]

Also \( V(x) \) is radially unbounded \( \Rightarrow \exists c \in \mathbb{R}: \) \( L_c = B_r \) and \( L_c \) compact \& t-rely inv. \( E = \{ x \in L_c \mid x = 0 \} \). This set is inv. since it is the eq. set. So \( M = E \).
From LaSalle's inv thm, trajectories starting in \( L_c \) approach \( E \) for any \( c \).

Note: \( V(x) \) has parameter \( b \) which need not be explicitly known, i.e., it may be possible to have existence of a desired \( V(x) \) without explicitly knowing it.

Linear systems & Linearizations

\[
i = Ax \text{ has isolated eq. at } 0 \text{ if det } A \neq 0 \quad \text{(general eq. set = null space of } A) \quad \text{Stability property of a linear system can be characterized using locations of eigenvalues}
\]
\[
i = \Lambda \text{ and } x(t) = e^{At} x(0) \quad \text{let } \rho^{-1} A = J = \text{block diag}[J_1, J_2, \ldots, J_r], \quad \text{where}
\]
\[
J_i \text{ is Jordan block associated with eigenvalue } \lambda_i \text{ of } A, \quad J_i = \begin{bmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 1 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & \lambda_i & 1 \\ 0 & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & 0 \\ \end{bmatrix}, \quad \text{of order } m_i
\]
\[
\Rightarrow e^{At} = \rho e^{J(t)} \rho^{-1} = \sum_{i=1}^{m} \sum_{k=1}^{m_i} \frac{t^k}{k!} e^{\lambda_i t} \text{Rik}
\]

If \( \lambda_i \) has multiplicity \( m_i \), then Jordan blocks associated with \( \lambda_i \) are all of order \( 1 \) if \( \text{rank}(A - \lambda_i I) = n - m_i \) (\( n \) is dimension of \( x \)).

Thm: \( x = 0 \) is stable eq. of \( x = Ax \) if \( \text{Re}(\lambda_i) \leq 0 \) and \( \text{Re}(\lambda_i) = 0 \Rightarrow \text{rank}(A - \lambda_i I) = n - m_i \).

\( x = 0 \) is globally asymp. stable if \( \text{Re}(\lambda_i) < 0 \).

Proof: \( 0 \) is stable if \( e^{At} \) bounded \( \forall t \geq 0 \). If \( \text{Re}(\lambda_i) > 0 \Rightarrow e^{At} \) cannot be bounded and so we must have \( \text{Re}(\lambda_i) \leq 0 \). If \( \text{Re}(\lambda_i) = 0 \Rightarrow e^{At} \) cannot be bounded if \( m_i \geq 2 \).
So we must have \( \text{Re}(\lambda_i) = 0 \Rightarrow \text{rank}(A - \lambda_i I) = n - m_i \). This establishes necessity.

Sufficiency follows from \( x(t) = e^{J(t)} x(0) \) and \( \text{Re}(J) = \sum_{k=1}^{m_i} \frac{t^k}{k!} \text{Rik} \). For asymp. stability, \( e^{At} \rightarrow 0 \) \( \forall t \rightarrow \infty \).
Linear system & Linearization (cont.)

\[ \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \]

For the given system, \( \rho = \pm j \) \( \Rightarrow \) stable.

For the series system, \( A_S = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \) and for parallel system, \( A_P = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \).

Then for \( A_S \) and \( A_P \), \( \rho = \pm j \) with \( q_i = 2 \) \( (\lambda_1 = j, \lambda_2 = -j) \).

Also, \( \text{rank} [A_P - \rho_i I] = n - q_i = 4 - 2 = 2 \), \( \text{rank} [A_S - \rho_i I] = 3 - n - q_i \).

Thus parallel connection stable, while series connection unstable.

In parallel connection, non-zero initial condition \( \Rightarrow \) const. amplitude osc. in both copies of the system. Sum of constant amp. osc. of same freq. \( \Rightarrow \) constant amp. osc.

In series connection, the const. amp. osc. of 1st copy excites the 2nd copy. Since the 2nd copy has a natural freq. of 1 rad/sec which is the freq. of driving input, "resonance" occurs and response grows unbounded.

A called Hurwitz if \( \text{Re}(\rho_i) < 0 \) for all \( i \).

Lyapunov method can be used to investigate asymptotic stability of \( \rho = Ax \).

Consider \( V(x) = x^T P x \) as a choice of Lyapunov fn. \( V > 0 \) \( \Rightarrow \) \( P > 0 \).

Also \( \dot{V}(x) = 2 x^T P \dot{x} = x^T P \dot{x} + 2 \dot{x}^T P x = x^T P A x + 2 \dot{x}^T P A x = z^T (A_P + PA) z \).

We want \( P \) s.t., \( A_P + PA = -Q \) for some \( Q > 0 \) to ensure asymptotic stability.

Thm: \( A \) is Hurwitz iff \( \forall Q > 0 \exists P > 0: A_P + PA = -Q \).

(\( \Leftarrow \)) Choose \( V(x) = x^T P x \).

(\( \Rightarrow \)) Let \( P = \int_0^\infty e^{A t} Q e^{A^T t} dt. \) Since \( e^{A t} = \sum \sum \sum \sum e^{A t} e^{A t} \) \( R(k) \) and \( \text{Re}(\rho_i) < 0 \), the integral exists and is finite.

To see \( P > 0 \), consider \( z^T P z = \int_0^\infty (z^T e^{A t} Q e^{A^T t} z) dt \) for \( z \neq 0 \).

Since \( Q > 0 \), \( z^T e^{A t} Q e^{A^T t} z > 0 \) for all \( t \) \( \Rightarrow \) \( \int_0^\infty (z^T e^{A t} Q e^{A^T t} z) dt > 0 \).

Finally, \( \dot{V}(x) = \int_0^\infty [A_P + PA] z^T Q e^{A^T t} e^{A t} z] dt = \int_0^\infty \frac{d}{dt} [z^T e^{A t} Q e^{A^T t} z] dt \)

\[ = e^{A t} Q e^{A^T t} z \bigg|_{t=0} - 0 - Q = -Q. \]

Now if \( \tilde{P} \) is another solution, i.e. \( -Q = A_P \tilde{P} + \tilde{PA} \). Then,

\[ P = \int_0^\infty (e^{A^T t} [A^T \tilde{P} + \tilde{PA}] e^{A t}) dt = \int_0^\infty \frac{d}{dt} [e^{A^T t} \tilde{P} e^{A t}] dt = e^{A^T t} \tilde{P} e^{A t} \bigg|_{t=0} = \tilde{P} - \tilde{P} \tilde{P} \tilde{P} = P. \]

Remark: \( Q \) can be chosen to be \( CTC \) \( (\Rightarrow Q > 0) \) such that \( (A, C) \) observable.
Since $Q$ can be chosen to be any positive definite matrix, one choice is $Q = I$.

**Example:**

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$.

$A^T P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} P_{12} & P_{22} \\ P_{21} & P_{11} \end{bmatrix}$

$P A = (A^T P)^T = \begin{bmatrix} P_{12} & -P_{22} + P_{11} \\ P_{22} & -P_{12} \end{bmatrix}$

$\Rightarrow A^T P + PA = \begin{bmatrix} 2 P_{12} & -P_{22} + P_{11} \\ P_{22} & -P_{12} \end{bmatrix}$

$\Rightarrow 2 P_{12} = -1 \Rightarrow P_{12} = -\frac{1}{2}$

$-P_{22} + P_{11} = 1 \Rightarrow P_{11} = 1 + P_{22}$

Thus, $P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}$.

Determinant:

$\det(P) = 1.25 > 0$ and $\det[Q] > 0$.

Checking whether $Re(\lambda_i) < 0$ is easier than determining $P > 0$ such as above.

The real advantage of finding $P > 0$ is in proving stability properties of linearization.

Given $x = f(x)$ with $f(0) = 0$, $f(x) = Ax + g(x)$, where

$A = \frac{df}{dx}\bigg|_{x=0}$ and $g(x) = \left[ \frac{df}{dx}(x) - \frac{df}{dx}(0) \right] x$.

Note from mean-value theorem,

$\frac{f(x)}{g(x)} = \frac{f(x) - f(0)}{g(x) - g(0)} = \frac{\frac{df}{dx}(x)}{\frac{df}{dx}(x) - \frac{df}{dx}(0)}$,

where $x$ lies in line from $0$ to $x$.

**Thm:** Linearly asymptotically stable if $Re(\lambda_i) < 0 \forall \lambda_i(A)$, and unstable if $Re(\lambda_i) > 0$ for some $\lambda_i(A)$, where $A = \frac{df}{dx}\bigg|_{x=0}$

For first part, use $V(x) = x^T P x$ as candidate. $V > 0 \Rightarrow P > 0$. Also

$\dot{V} = 2 x^T P \dot{x} = 2 x^T Pa + 2 x^T P \dot{x} = 2 x^T (P A + A^T P) x$

$= 2 x^T [P A + A^T P] x + 2 x^T P g = -2 x^T Q x + 2 x^T P g$

$\Rightarrow \dot{V} < 0 \forall x$.

So, $\dot{V} < -2 x^T Q x + 2 x^T P g < 0$. Also $z^T Q z > \lambda_{\text{min}}(Q) \| z \|^2 \Rightarrow \dot{z} < -[\lambda_{\text{min}}(Q) - 2 y] \| z \|^2$. Thus choosing $\gamma < \frac{1}{2} \lambda_{\text{min}}(Q) / \gamma_{\text{ell}}$ ensures $\dot{V} < 0$.

For 2nd part, first suppose $Re(\lambda_i) > 0 \forall \lambda_i$. By defining $z = T x$, we can have $T A T^{-1} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ with $A_1, A_2 > 0$ and

$z_i = z_i + \gamma_i (z_i) \Rightarrow \dot{z} = A_1 z_1 + A_2 z_2$

$z_2 = A_2 z_2 + \gamma_i (z_i) \Rightarrow \| z \|^2 + \lambda_{\text{min}}(Q) \| z \|^2 \Rightarrow \dot{V} > 0$.

Thus $\lambda_{\text{min}}(Q) > 0$ for $Q > 0$.

Define $V(x) = \epsilon_1 x_1^2 + \epsilon_2 x_2^2 = \epsilon_1 \left[ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right] x = V(x) > 0$ for $z = 0$.

Define $V(x) = \epsilon_1 x_1^2 + \epsilon_2 x_2^2 = \epsilon_1 \left[ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right] x = V(x) > 0$ for $z = 0$. 

$A > 0 \Rightarrow Re(\lambda_i) > 0 \forall \lambda_i \Rightarrow \dot{V} > 0$. 

Thus, $P A + A^T P = -Q_c$.

Define $V(x) = \epsilon_1 x_1^2 - \epsilon_2 x_2^2 = \epsilon_1 \left[ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right] x = V(x) > 0$ for $z = 0$,

Define $V(x) = \epsilon_1 x_1^2 + \epsilon_2 x_2^2 = \epsilon_1 \left[ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right] x = V(x) > 0$. 

$\epsilon_1 > 0 \Rightarrow Re(\lambda_i) < 0 \forall \lambda_i \Rightarrow \dot{V} < 0$. 

Thus, $P A + A^T P = -Q_c$.
Linearization (contd.)

\[ v(z) = 2z_1^T p_1 z_1 - 2z_2^T p_2 z_2 = z_1^T p_1 z_1 + z_2^T p_2 z_2 - z_2^T p_2 z_2 \]

\[ = (-A_1 z_1 + g_1)^T p_1 z_1 + z_1^T p_1 (-A_1 z_1 + g_1) - (A_2 z_2 + g_2)^T p_2 z_2 - z_2^T p_2 (A_2 z_2 + g_2) \]

\[ = z_1^T \left( A_1^T p_1 + p_1 A_1 \right) z_1 + 2z_1^T p_1 g_1 - z_2^T \left( A_2^T p_2 + A_2 p_2 \right) z_2 - 2z_2^T p_2 g_2 \]

\[ = z_1^T q_1 z_1 + z_2^T q_2 z_2 + 2z_1^T \left[ p_1 g_1 \right] \]

\[ \geq \lambda_{\min}(Q_1) ||z_1||^2 + \lambda_{\min}(Q_2) ||z_2||^2 - 2 ||z_1|| \sqrt{||p_1||^2 ||g_1||^2 + ||p_2||^2 ||g_2||^2} \]

\[ > \alpha (||z_1||^2 + ||z_2||^2) - 2 ||z_1|| \sqrt{\beta^2 ||z_1||^2 + ||z_2||^2} \]

\[ \alpha = \min \left( \lambda_{\min}(Q_1), \lambda_{\min}(Q_2) \right), \beta = \max \left( ||p_1||, ||p_2|| \right) \]

\[ = (\alpha - 2\sqrt{\beta}) ||z||^2. \]

Thus choosing \( \gamma < \frac{\alpha}{2\sqrt{\beta}} \) ensures \( v > 0 \). Unstability follows from Thm 4.3. When \( \text{Re}(\lambda_i) = 0 \) for some \( \lambda_i \), then let \( S = \min_{\lambda_i} \text{Re}(\lambda_i) > 0. \)

Then \( (A - \frac{S}{2} I) \) is such that \( \text{Re}(\lambda_i (A - \frac{S}{2} I)) \neq 0. \) From previous analysis, for \( Q > 0 \) exists \( P \) s.t. \( P \left[ A - \frac{S}{2} I \right] + \left[ A - \frac{S}{2} I \right]^T P = Q > 0 \) and \( v(z) = z^T P z \) is positive for points arbitrarily close to \( 0. \)

Also, \( v(z) = 2z_1^T P z_1 + z_2^T P z_2 = 2z_1^T \left[ A^T P + P A \right] z_1 + 2z_1^T P g_1 \]

\[ = 2z_1^T \left( A - \frac{S}{2} I \right)^T P + P (A - \frac{S}{2} I) \] z_1 + 8z_1^T P z_2 + 2z_1^T P g_1 \]

In the set, \( \{ z \in \mathbb{R}^n \mid ||z|| \leq \gamma \} \), where \( \gamma \) is chosen so that \( ||z|| \leq \gamma \) and \( v > 0 \), we have

\[ v \geq \lambda_{\min}(Q) ||z||^2 - 2 \gamma \|P\| ||z||^2 = (\gamma \lambda_{\min}(Q) - 2 \gamma ||P||) ||z||^2, \]

which is true if

\[ \gamma < \frac{\lambda_{\min}(Q)}{2 ||P||}. \]

Unstability follows from Thm 4.3.

**Remark:** Stability property of nonlinear system can be deduced from its linearization provided \( \text{Re}(\lambda_i) \neq 0. \) The test is based on computation of \( \text{Re}(\lambda_i) \) and checking its location, whereas proof is based on Lyapunov.