Sampling

- Sampling used to convert cont.-time signal into a discrete-time signal

\[ x(t) \rightarrow \hat{x}(t) = x(t) \delta(t) \]

- Ideal Sampler: \( p(t) = S_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) \)

\[ x^*(t) = x(t) \delta(t) = \sum_{n=-\infty}^{\infty} p(t-nT) = x(t) \sum_{n=-\infty}^{\infty} p(t-nT) \]

\[ x^*(t) = \sum_{n=-\infty}^{\infty} x(nT) p(nT) e^{-nTA} \]

For ideal sampler, \( p(t) = S_T(t) \Rightarrow p(s) = 1 \)

So for ideal sampler, \[ X^*(s) = \sum_{n=0}^{\infty} x(nT) e^{-nTA} \]

Thus \[ X^*(s) = X(z) \bigg|_{z=e^{-sT}} \]

Example: \[ X(s) = \frac{1}{(sA+1)(sA+2)} = \frac{1}{sA+1} \frac{1}{sA+2} \Rightarrow x(t) = x(t) \delta(t) \]

\[ x(nT) = (e^{-nT} - e^{-2nT}) u(nT) \]

\[ X(z) = \frac{z}{z-e^{-T}} - \frac{z}{z-e^{-2T}} \]

\[ X^*(s) = \frac{e^{sT}}{e^{sT}-e^{T}} - \frac{e^{sT}}{e^{sT}-e^{2T}} = \frac{1}{1-e^{-T(s+1)}} - \frac{1}{1-e^{-T(sA+2)}} \]
Sampling

(i) \( X^*(s) = \sum_{n=0}^{\infty} x(nt) e^{-nt} \) is periodic with period \( j2\pi/T \).

\[
x^*(s+j2\pi/T) = \sum_{n=0}^{\infty} x(nt) e^{-nt}(s+j2\pi) = \sum_{n=0}^{\infty} x(nt) e^{-nt} e^{-j2\pi n} = \sum_{n=0}^{\infty} x(nt) e^{-nt} \]

(ii) poles & zeros of \( X^*(s) \) are periodic with period \( j2\pi/T \).
In particular, if \( X(s) \) has pole at \( s_1 \), then \( X^*(s) \) has
poles at \( s_1 + jn(2\pi/T) \). There is no such direct relation
between zeros of \( X(s) \) and \( X^*(s) \).

\[
x^*(t) = x(t) p_1(t) = x(t) \left[ \sum_{n=-\infty}^{\infty} p_n e^{jn(2\pi/T)t} \right]
\]

\[
\Rightarrow X^*(s) = \sum_{n=-\infty}^{\infty} p_n X(s-jn2\pi/T)
\]

Note: \( p_1(t) = \sum_{t=0}^{T} S(t) e^{-j\pi t} \Rightarrow p_n = \frac{1}{T} \int_{0}^{T} S(t) e^{-j2\pi nt/T} \, dt = \frac{1}{T} \).

If \( X(s) \) has pole at \( s_1 \), then each term \( X(s-jn2\pi/T) \) contributes
a pole at \( s_1 + jn2\pi/T \).

\[
(X(s) = \frac{1}{s} \Rightarrow X^*(s) = \frac{p_0}{s} + \frac{p_1}{s-j2\pi/T} + \frac{p_2}{s-j4\pi/T} + \cdots
\]

\[
= \frac{\delta(s-j2\pi/T)(s-j4\pi/T) \cdots}{\delta(s-j2\pi/T)(s-j2\pi/T) \cdots}
\]

\( \Rightarrow \) poles at \( s=0, j2\pi/T, j4\pi/T \cdots \)

(iii) Spectrum of \( X^*(s) \), namely \( X^*(jw) \), is “periodic” with period \( 2\pi/T \).

\[
X^*(jw) = \sum_{n=-\infty}^{\infty} p_n X(j(w-n2\pi/T)) = \sum_{n=-\infty}^{\infty} p_n X(j(w-n^{2\pi/T}))
\]

\[
|X(j\omega)| \quad \omega_0 < 2\pi/T - \omega_0 = \frac{1}{T} \quad \omega_0/\pi = 2f_0
\]
Sampling

- A consequence of above spectrum of $X^\ast(j\omega)$ is that spectrum of $X(j\omega)$ can be **uniquely** recovered by way of low-pass filtering provided $X^\ast(j\omega)$ is band-limited (no freq. components greater than some freq. $\omega_0$) and sampling occurs at least at rate $T = \frac{T_0}{2\omega_0}$ ("sampling rate $\geq 2$ bandwidth" $\Leftarrow$ Shannon's Sampling Thm).

$X^\ast(\lambda) = \sum \frac{\text{Residue of } X(\lambda)}{1 - e^{-T(\lambda - \lambda)}}$

$\lambda \in \text{poles of } X(\lambda)$

**Example:**

$X(\lambda) = \frac{1}{(\lambda+1)(\lambda+2)}$

$X^\ast(\lambda) = \frac{1}{(\lambda+2)(1-e^{-T(\lambda - \lambda)})} + \frac{1}{(\lambda+1)(1-e^{-T(\lambda - \lambda)})} \bigg|_{\lambda=1} \bigg|_{\lambda=-2}$

$= \frac{1}{1-e^{-T(\lambda+1)}} - \frac{1}{1-e^{-T(\lambda+2)}}$

Reconstruction

- Original signal can be reconstructed from sampled signal through low pass filtering provided sampling rate exceeds twice the signal bandwidth. Data holders essentially act as low pass filters.

If sampling rate does not exceed twice the maximum frequency in the signal, **aliasing** occurs, and low pass filtering will recover the original signal together with some distortion.