**Current observer**

- The observer we discussed earlier doesn't use the most recent observation $y(k+1)$ in estimating $\hat{x}(k+1)$. We can build a "current observer" as follows:

  \[
  \begin{align*}
  \text{predict:} & \quad \tilde{x}(k+1) = A \hat{x}(k) + B u(k) \\
  \text{update:} & \quad \hat{x}(k+1) = \tilde{x}(k+1) + G \left[ y(k+1) - C \tilde{x}(k+1) \right]
  \end{align*}
  \]

  Assumed $D = 0$

  Thus \( \hat{x}(k+1) = A \hat{x}(k) + Bu(k) + G \left[ y(k+1) - C A \hat{x}(k) - Bu(k) \right] \)

  \( = (A - GCA) \hat{x}(k) + (B - GCB) u(k) + Gy(k+1). \)

  \( \tilde{x}(k+1) = x(k+1) - \hat{x}(k+1) \)

  \( = [A \hat{x}(k) + Bu(k)] - [(A - GCA) \hat{x}(k) + (B - GCB) u(k) + Gy(k+1)] \)

  \( = (A - GCA) \tilde{x}(k) \)

  So $G$ must be chosen so that $(A - GCA)$ is stable. For a desired char. poly. $Q(z)$, we can choose $G$ as:

  \[
  G = Q(z) \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^n \\ 1 \end{bmatrix} \quad (\text{requires observability } P(A,CA)).
  \]

**Kalman Filter**

System model subject to disturbance and noise:

\[
\begin{align*}
  x(k+1) &= A x(k) + B u(k) + W(k) \quad W(k) \sim N(0, R_w(k)) \\
  y(k) &= C x(k) + V(k) \quad V(k) \sim N(0, R_v(k)) \\
  R_w(k) &= E[W(k)W^T(k)] \text{ is the covariance matrix of } W(k) \text{. Similarly } R_v(k).
\end{align*}
\]

It is assumed that \{\(W(k), V(k), x(k)|k=0,1,...\} are mutually independent.

Kalman filter is an optimal current observer:

\[
\begin{align*}
  \tilde{x}(k+1) &= A \hat{x}(k) + B u(k) \\
  \hat{x}(k+1) &= \tilde{x}(k+1) + G(k)[y(k+1) - C \tilde{x}(k+1)], \quad \text{where} \\
  G(k) \text{ is chosen to minimize } E[e(k)^T e(k)] = E[(x(k) - \hat{x}(k))^T(x(k) - \hat{x}(k))].
\end{align*}
\]
Note \( \mathbb{E}[e^T(k+1)e(k+1)] = \mathbb{E}[e^T(k+1)e^T(k+1)] = \mathbb{E}(R_e(k+1)). \)

We derive \( R_e(k+1) = \text{cov}(e(k+1)) = \text{cov}(\hat{z}(k+1) - \tilde{z}(k+1)). \)

\[
R_e(k+1) = \text{cov}(\tilde{z}(k+1)) = \text{cov}(z(k+1) - \tilde{z}(k+1))
\]

\[
R_e(k+1) = \text{cov}[z(k+1) - \tilde{z}(k+1)]
\]

\[
= \text{cov}[z(k+1) - \tilde{z}(k+1) + G(k+1) \tilde{z}(k+1)]
\]

\[
= \text{cov}[z(k+1) - \tilde{z}(k+1) + G(k+1) \tilde{z}(k+1)]
\]

\[
= \text{cov}[z(k+1) - \tilde{z}(k+1) + G(k+1) \tilde{z}(k+1)]
\]

\[
= \text{cov}[z(k+1) - \tilde{z}(k+1) + G(k+1) \tilde{z}(k+1)]
\]

\[
= \text{cov}[(I - G(k+1)C)(z(k+1) - \tilde{z}(k+1))] + \text{cov}[G(k+1) \tilde{z}(k+1)]
\]

\[
= (I - G(k+1)C) R_e(k+1) (I - G(k+1)C)^T + G(k+1) R_e(k+1) G(k+1)^T
\]

\[
= R_e(k+1) - G(k+1) C R_e(k+1) - G(k+1) C^T G(k+1)^T
\]

\[
+ G(k+1) C R_e(k+1) C^T G(k+1) + G(k+1) R_e(k+1) G(k+1)^T
\]

To minimize \( \text{tr} [R_e(k+1)] \), we set

\[
0 = \frac{\partial}{\partial G(k+1)} \text{tr} [R_e(k+1)]
\]

\[
= -2 R_e(k+1) C^T + 2 G(k+1) C R_e(k+1) C^T + 2 G(k+1) R_e(k+1) C^T
\]

\[
= G(k+1) = R_e(k+1) C^T [R_e(k+1) C^T + R_e(k+1)]^{-1}.
\]

Also at above \( G(k+1) \), \( R_e(k+1) = R_e(k+1) - G(k+1) C R_e(k+1) \)

(since last two terms in \( R_e(k+1) \) cancel) = \( [I - G(k+1) C] R_e(k+1) \)

Also, \( R_e(k+1) = \text{cov} [z(k+1) - \tilde{z}(k+1)] = \text{cov} [A z(k) + B u(k) + \nu(k) - A \hat{z}(k) B u(k)] \)

\[
= \text{cov} [A (z(k) - \hat{z}(k)) + \nu(k)]
\]

\[
= A R_e(k) A^T + R_u(k).
\]
Kalman Filter

**Predict:**
\[
\hat{\mathbf{x}}(k+1) = A \hat{\mathbf{x}}(k) + B u(k)
\]
\[
R_{\hat{\mathbf{e}}}(k+1) = A R_{\hat{\mathbf{e}}}(k) A^T + R_w(k)
\]

**Update:**
\[
\hat{\mathbf{x}}(k+1) = \hat{\mathbf{x}}(k+1) + G(k+1) \left[ y(k+1) - C \hat{\mathbf{x}}(k+1) \right]
\]
\[
R_{\hat{\mathbf{e}}}(k+1) = \left[ I - G(k+1)C \right] R_{\hat{\mathbf{e}}}(k+1)
\]
\[
G(k+1) = R_{\hat{\mathbf{e}}}(k+1) C^T \left[ C R_{\hat{\mathbf{e}}}(k+1) C^T + R_v(k+1) \right]^{-1}
\]

where
\[
R_{\hat{\mathbf{e}}}(k+1) = \text{cov} \left[ \hat{\mathbf{e}}(k+1) \right] = \text{cov} \left[ \mathbf{z}(k+1) - \hat{\mathbf{x}}(k+1) \right]
\]
\[
R_{\hat{\mathbf{e}}}(k+1) = \text{cov} \left[ e(k+1) \right] = \text{cov} \left[ \mathbf{z}(k+1) - \hat{\mathbf{z}}(k+1) \right].
\]

To initialize we may choose \( \hat{\mathbf{x}}(0) = 0 \Rightarrow R_{\hat{\mathbf{e}}}(0) = \text{cov} \left[ \mathbf{z}(0) - \hat{\mathbf{z}}(0) \right] = \text{cov} \left[ \mathbf{z}(0) \right]. \)

**Example:**
\[
\mathbf{x}(k+1) = \begin{bmatrix} 1 & .0952 \\ 0 & .905 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} .0048 & 0 \\ .0952 & 0 \end{bmatrix} \mathbf{u}(k) + \mathbf{w}(k)
\]
\[
\mathbf{y}(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{z}(k) + \mathbf{v}(k)
\]
\[
R_u(k) = \text{diag} \left[ \sigma_u^2, \sigma_u^2 \right], \quad R_v(k) = \text{diag} \left[ \sigma_v^2, \sigma_v^2 \right], \quad \mathbf{R}_u = \mathbf{R}_v = \mathbf{1}
\]

Assuming \( R_{\hat{\mathbf{e}}}(0) = \mathbf{I} \) and

Using simulation, \( G = 0.5G(k) = \begin{bmatrix} .636 \\ .570 \end{bmatrix} \)

(convergence occurs in 15 iterations)

Also \( R_{\hat{\mathbf{e}}}(k) \) converged to \( \begin{bmatrix} .636 & .570 \\ .570 & .575 \end{bmatrix} \), which means

variance of error in \( \mathbf{z}_1 \) is \( .636 = (745)^2 \) and in \( \mathbf{z}_2 = .570 = (745)^2 \)

Thus if actual measurement of \( \mathbf{z}_1 \) is 7, then real value of \( \mathbf{z}_2 \) lies within \( 7 \pm 3(745) \) with 99.7% prob.

(\( \text{Aside the s.d. covers to 99.7% of distribution.} \))