Many times system specs are given via s-poles as they determine time-const. (= \text{inv} |Re(pole)|) and damping (= \text{inv} \tan |Im(pole)/Re(pole)|). Dom time-const. = nearest to im-axis; dom damping = biggest inclination to im-axis; Q: Relationship to z-poles?

**Mapping s-plane into z-plane**

\[ z = e^{sT} \text{ maps } s \text{ to } z. \]

- **Imaginary axis** : \( s = j\omega, \omega \in [-\infty, \infty] \)
  \[ \Rightarrow z = e^{sT} = e^{j\omega T} = 1 / \omega T \leftrightarrow \text{unit circle} \]

- **LHP**: \( s = \sigma + j\omega, \sigma < 0, \omega \in \mathbb{R} \)
  \[ \Rightarrow z = e^{sT} = e^{(\sigma + j\omega)T} = e^{\sigma T} e^{j\omega T} \leftrightarrow \text{inside of unit circle} \]
  \[ \sigma < 0 \Rightarrow e^{\sigma T} < 1 \]

- **RHP**: \( s = \sigma + j\omega, \sigma > 0, \omega \in \mathbb{R} \)
  \[ \leftrightarrow \text{outside of unit circle} \]

**Stability property determined by locations of transfer fn. poles (= roots of characteristic equation)**
- PTF stable if all poles inside unit circle
- PTF marginally stable if no poles outside unit circle and poles on unit circle have multiplicity one.

- **Constant damping**: \( s = \sigma_0 + j\omega_0, \omega_0 \in [-\infty, \infty] \)
  \[ \Rightarrow z = e^{sT} = e^{\sigma_0 T} / \omega_0 \leftrightarrow \text{circle with radius } e^{\sigma_0 T} \]

- **Constant freq.**: \( s = \sigma + j\omega_0, \sigma_0 \in [-\infty, \infty] \)
  \[ \Rightarrow z = e^{sT} = e^{\sigma T} / \omega_0 \leftrightarrow \text{ray with angle } \sigma_0 T \]
Mapping \( z \)-poles to time constant, damping ratio, natural freq.

- Simple pole at \( z = r/e^j\theta = re^{j\theta} \)
  \( \Rightarrow \) pole at \( e^{\delta T} = r e^{j\theta} \)  \( \Rightarrow \) pole at \( \delta = \ln r + j\theta \)
  \( \Rightarrow \) pole at \( s = \frac{\ln r + j\theta}{T} \)  \( \Rightarrow \) time const. = \( \left\lfloor \frac{T}{\ln r} \right\rfloor \)
  (Time constant is inverse of mag. of real part of pole)

- Complex conjugate poles at \( z = r/e^{\pm j\theta} = re^{\pm j\theta} \)
  \( \Rightarrow \) pole at \( e^{\delta T} = r e^{\pm j\theta} \)  \( \Rightarrow \) pole at \( \delta = \frac{\ln r \pm j\theta}{T} \)
  \( \Rightarrow \) Time const. = \( \frac{1}{\ln r} \) and characteristic eq. in \( s \)-plane:
    \[ (s - \left( \frac{\ln r + j\theta}{T} \right)) [s - \left( \frac{\ln r - j\theta}{T} \right)] = 0 \]
  \( \Rightarrow (s - \frac{\ln r}{T})^2 + (\frac{\theta}{T})^2 = 0 \)
  \( \Rightarrow s^2 - 2\frac{\ln r}{T} s + \frac{\ln r}{T} + \frac{\theta^2}{T^2} = 0 \)
  So, \( \omega_n = \sqrt{\frac{\ln r}{T} + \frac{\theta^2}{T}} \); \( 2\zeta \omega_n = -2 \frac{\ln r}{T} \)
  \( \Rightarrow \zeta = -\frac{\ln r}{\omega_n T} = \frac{-\ln r}{\sqrt{\ln^2 r + \theta^2}} \)

Example: \( G(z) = \frac{368.3 + 264}{z^2 - 2 + 632} \), \( T = 1 \)
\( z^2 - 2 + 632 = 0 \)  \( \Rightarrow \) \( z = 0.5 \pm j0.618 = 0.795 \pm 0.89 \)
\( \Rightarrow \) time const. = \( \left\lfloor \frac{1}{\ln 0.795} \right\rfloor = 4.36 \text{ sec.} \)
\( \omega_n = \sqrt{\frac{\ln 0.795}{T} + \frac{0.89^2}{T}} = 0.9191 \text{ rad/sec.} \)
\( \zeta = \frac{-\ln 0.795}{\sqrt{\ln^2 0.795 + 0.89^2}} = 0.25 \)

Compare this to cont. time tf: \( G(s) = \frac{1}{s^2 + 1} \)  \( \Rightarrow \zeta = 0.5 \)
\( \omega_n = 1 \)  \( \Rightarrow T = 2 \text{ sec.} \)
Mapping z-poles to time-constant

The set of parameters \((z, \bar{z}, \omega_n)\) are quite different. This is because sampling period is only half of time-constant. We need \(T \ll z\), and also \(\left(\frac{\omega_n}{\sqrt{1-\zeta^2}}\right) T \ll 1\) for oscillation.

Choose \(T = 0.1\), then can be shown that for discrete-time system \(z = 2.11, \bar{z} = 0.475, \omega_n = 0.998\). \([\text{HW problem}]\)

For stable system, \(\rho < 1 \Rightarrow z = \frac{-T}{\ln \rho} \Rightarrow \rho = e^{-\frac{T}{\ln z}}\)

\(\theta/T\) is imaginary part, and equals \(\omega_n \sqrt{1-\zeta^2}\) (whereas \(\ln(\rho)/T\) is real part that equals \(\omega_n\zeta\)).

So if \(z, \omega_n, \zeta\) are known, we can obtain \(\rho\) and \(\theta\).

Info about sys. response is contained in impulse response, or equivalently step-response (= integral of impulse-response since step is integral of impulse), and so sys. specs may be given as step-response parameters.

Step-response of simple real pole (1st order): \(1 - e^{-\sigma t}\);
Step-response of simple complex pole (2nd order under-damped \(\Rightarrow \zeta < 1\)): \(1 - (1/\sqrt{1-\zeta^2}) e^{-\sigma t} \sin(\omega_d t + \cos^{-1} \zeta); \sigma = \omega_n \zeta; \omega_d = \omega_n \sqrt{1-\zeta^2}\).
Sys. specs may be given as peak-overshoot, rise-time, settling-time:
Peak-overshoot at half-period \(\pi/\omega_d = \pi/(\omega_n \sqrt{1-\zeta^2})\), with value \(e^{-\pi \zeta / \sqrt{1-\zeta^2}}\);
Rise-time (time to reach value 1) is \((\pi - \cos^{-1} \zeta)/\omega_d\);
Settling time is three to five time-constants: \(3/\omega_n \zeta\) to \(5/\omega_n \zeta\).