Branch & Bound for Combinatorial Optimization

- Combinatorial Optimization = Decision variables integer & bounded
- Such problems can be converted to BP
  Suppose \( x_i < 2^{n+1} \), then replace \( x_i = \sum_{k=0}^{n} 2^k q_k^{(i)} \)
  Example: \( x_i < 32 = 2^{4+1} \) \( \Rightarrow x_i = 16 q_4^{(i)} + 8 q_3^{(i)} + 4 q_2^{(i)} + 2 q_1^{(i)} + q_0^{(i)} \)

- So we study branch & bound for LBP.
  - No need to analyze "LP relaxations", rather analyze "LBPs".
  - Root node: Given LBP; Branching occurs on \( (x_i = 0) \) or \( (x_i = 1) \)

- n binary decision variables \( \Rightarrow \) tree can have up to \( 2^n \) nodes
- At P1, all variables are "free";
- At P2/P3, \( x_1 \) is fixed, others are free;
- At P4/P5/P6/P7, \( x_1, x_2 \) are fixed, others are free.

- At a node "free variables" are available for assignment; they can be assigned for "best completion" cost a constraint/objective (BC)

\[
\min z = 7x_1 + 3x_2 + 2x_3 + x_4 + 2x_5
\]

\[
st. \quad 4x_1 + 2x_2 - x_3 + 2x_4 + x_5 \geq 3 \quad (C1)
\]

\[
4x_1 + 2x_2 + 4x_3 - x_4 - 2x_5 \geq 7 \quad (C2)
\]

- At P6, \( x_1 = 1 \), \( x_2 = 0 \) fixed, \( x_3, x_4, x_5 \) "free". BC wot \( z = 2 + 2 + 0 = 4 \)
  BC wot C1: \( x_3 = 0, x_4 = x_5 = 1 \); BC wot C2: \( x_3 = 1, x_4 = x_5 = 0 \).

**At a node branch on a free variable unless**

(i) BC wrt any constraint infeasible,
(ii) BC wrt $z$ is inferior to current bound (initial bound = $-\infty$ for max and $\infty$ for min)
(iii) BC wrt $z$ is also feasible.
(Stop, when no branching possible $\Rightarrow$ Current bound is optimal)

**Example:**

\[
\begin{align*}
\text{min } z &= 7x_1 + 3x_2 + 2x_3 + x_4 + 2x_5 \\
\text{s.t.} & \\
4x_1 + 2x_2 - x_3 + 2x_4 + x_5 & \geq 3 \quad (C1) \\
4x_1 + 2x_2 + 4x_3 - 2x_4 - 2x_5 & \geq 7 \quad (C2) \\
x_1 &= 0 \text{ or } 1 \\
\end{align*}
\]

**Iteration 1:**

bound = $\infty$. $P_1 =$ Given LBP $\Rightarrow$ all variables free.

(i) BC wrt $C_1$: $x_1 = x_2 = x_4 = x_5 = 1$, $x_3 = 0 \Rightarrow$ LHS($C_1$) = 9 $\geq 3$ not inf.

(ii) BC wrt $C_2$: $x_1 = x_2 = x_3 = 1$, $x_4 = x_5 = 0 \Rightarrow$ LHS($C_2$) = 10 $\geq 7$ inf.

(iii) BC wrt $z$: $x_1 = x_2 = x_3 = x_4 = x_5 = 0$, which is infeasible (violates $C_1$ & $C_2$)

$\Rightarrow$ branch on a free variable, say $x_1$

$P_2 = P_1 + (x_1 = 0)$; $P_3 = P_1 + (x_1 = 1)$.

**Iteration 2:**

bound = $\infty$. $P_2 \Rightarrow x_1 = 0$ is fixed; $x_2$, $x_3$, $x_4$, $x_5$ free.

(i) BC wrt $C_1$: $x_2 = x_4 = x_5 = 1$, $x_3 = 0 \Rightarrow$ LHS($C_1$) = 5 $\geq 3$

(ii) BC wrt $C_2$: $x_2 = x_3 = 1$, $x_4 = x_5 = 0 \Rightarrow$ LHS($C_2$) = 6 $\neq 7$

BC wrt $C_2$ infeasible $\Rightarrow$ no further branching at $P_2$. 
Branch & Bound for LBP

\[
\begin{align*}
\min z &= 7x_1 + 3x_2 + 2x_3 + x_4 + 2x_5 \\
\text{subject to} & \\
& 4x_1 + 2x_2 - x_3 + 2x_4 + x_5 \geq 3 \\
& 4x_1 + 2x_2 + 4x_3 - x_4 - 2x_5 \geq 7
\end{align*}
\]

**Iteration 3:** Bound = \infty. \ P3 = 0 \ x_1 = 1 \text{ is fixed}; \ x_2, x_3, x_4, x_5 \text{ free}

(i) BC wrt \ x_4 = 0 \text{ same as \ P1} \ (\text{NOT infeasible})

(ii) BC wrt \ l = \{x_2 = x_3 = x_4 = x_5 = 0 = 0; \ \text{LHS}(c_1) = 4 \geq 3; \ \text{LHS}(c_2) = 4 \geq 7\}

= \text{branch on a free variable, say \ x_2}

\[\begin{align*}
P_4 &= P_3 + (x_2 = 0) \quad \text{or} \quad P_5 = P_3 + (x_2 = 1)
\end{align*}\]

**Iteration 4:** Bound = \infty. \ P4 = \{
\begin{align*}
& x_1 = 1, \ x_2 = 0 \text{ fixed}; \ x_3, x_4, x_5 \text{ free} \\
& (i) \text{ BC wrt } c_1: x_3 = 0, x_4 = x_5 = 1 \Rightarrow \text{LHS}(c_1) = 7 \geq 3 \ \checkmark \text{ not infeasible} \\
& \text{BC wrt } c_2: x_3 = 1, x_4 = x_5 = 0 \Rightarrow \text{LHS}(c_2) = 8 \geq 7 \ \checkmark \text{ infeasible}
\end{align*}\]

(ii) BC wrt \ l = \{x_3 = x_4 = x_5 = 0 = 0; \ \text{LHS}(c_1) = 4 \geq 7; \ \text{LHS}(c_2) = 4 \geq 7\}

= \text{branch on a free variable, say \ x_3}

\[\begin{align*}
P_6 &= P_4 + (x_3 = 0) \quad \text{or} \quad P_7 = P_4 + (x_3 = 1)
\end{align*}\]

**Iteration 5:** Bound = \infty. \ P6 = \{
\begin{align*}
& x_1, x_2, x_3 = 0 \text{ fixed}; \ x_4, x_5 \text{ free} \\
& (i) \text{ BC wrt } c_1: x_4 = x_5 = 1 \Rightarrow \text{same as P4} \\
& \text{BC wrt } c_2: x_4 = x_5 = 0 \Rightarrow \text{LHS}(c_2) = 4 \geq 7 \ \checkmark \text{ infeasible}
\end{align*}\]

So no further branching at \ P6.

**Iteration 6:** Bound = \infty. \ P7 = \{
\begin{align*}
& x_1 = 1, x_2 = 0, x_3 = 1 \text{ fixed}; \ x_4, x_5 \text{ free} \\
& (i) \text{ BC wrt } c_1: x_4 = x_5 = 1 \Rightarrow \text{LHS}(c_1) = 6 \geq 3 \ \checkmark \text{ not infeasible} \\
& \text{BC wrt } c_2: x_4 = x_5 = 0 \Rightarrow \text{LHS}(c_2) = 8 \geq 7 \ \checkmark \text{ infeasible}
\end{align*}\]

(ii) BC wrt \ l = \{x_4 = x_5 = 0 = 0; \ \text{LHS}(c_1) = 3 \geq 3; \ \text{LHS}(c_2) = 8 \geq 7 \ \checkmark \text{ feasible!}

= \text{No further branching at \ P7; new bound = 9}
Branch and Bound for LBP

Given LBP
\[\begin{align*}
\min \ z &= x_1 + 3x_2 + 2x_3 + x_4 + 2x_5 \\
\text{st.} \ 4x_1 + 2x_2 - x_3 + 2x_4 + x_5 &\geq 3 \quad (C_1) \\
4x_1 + 2x_2 + 4x_3 - x_4 - 2x_5 &\geq 7 \quad (C_2)
\end{align*}\]

Iteration 7: Bound = 9. P5 \(\Rightarrow\) \(x_1 = x_2 = 1\) fixed; \(x_3, x_4, x_5\) free.

(i) BC w/ C1: \(x_3 = 0, x_4 = x_5 = 1 \Rightarrow LHS(C1) = 9 \geq 3\) \(\Rightarrow\) infeasible

(ii) BC w/ C2: \(x_3 = 1, x_4 = x_5 = 0 \Rightarrow LHS(C2) = 10 \geq 7\)

\(\Rightarrow\) no further branching at P5.

Optimal \(z = 9;\) \(x_1 = 1, x_2 = 0, x_3 = 1, x_4 = x_5 = 0.\)