The Envelope Theorem

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The unconstrained case

Consider a simple maximization problem

$$\max_x f(x, \theta)$$  (1)

where $x$ is a choice variable and $\theta$ is a parameter that we do not control. We assume that $f$ is a concave function of $x$ so that the first order conditions are not only necessary but sufficient for maximization.
An interior solution of (1) satisfies the following first order conditions:

\[
\frac{\partial f}{\partial x}(x, \theta) = 0 \rightarrow x^*(\theta)
\]  

Let’s plug this optimal value into the objective function and define the optimal value of the objective function as a function of \( \theta \).

\[
V(\theta) \equiv f(x^*(\theta), \theta).
\]
The question

We are interested in what happens to the optimal value of $f$ when the parameter $\theta$ changes. Formally, we are interested in

$$\frac{dV}{d\theta}(\theta).$$

Note that $V$ will change both because $\theta$ affects $f$ and because it also affects the optimal choice of $x$. 
To answer this question we take derivatives:

\[ \frac{dV}{d\theta}(\theta) = \frac{\partial f}{\partial x}(x^*(\theta), \theta) \frac{\partial x^*}{\partial \theta}(\theta) + \frac{\partial f}{\partial \theta}(x^*(\theta), \theta) \]
To answer this question we take derivatives:

\[
\frac{dV}{d\theta}(\theta) = \left. \frac{\partial f}{\partial x}(x^*(\theta), \theta) \right|_{0} \frac{\partial x^*}{\partial \theta}(\theta) + \frac{\partial f}{\partial \theta}(x^*(\theta), \theta)
\]
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\]

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To answer this question we take derivatives:

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\frac{dV}{d\theta}(\theta) = \frac{\partial f}{\partial x}(x^*(\theta), \theta) \frac{\partial x^*}{\partial \theta}(\theta) + \frac{\partial f}{\partial \theta}(x^*(\theta), \theta) \tag{0}
\]

\[
= \frac{\partial f}{\partial \theta}(x^*(\theta), \theta)
\]

where the second equality follows from (2).
This proves the simple version of the envelope theorem: the total rate of change in the optimal value of the objective function due to a small change in the parameter $\theta$ is simply the rate of change in the objective function $f$ evaluated at the optimal value of $x$.

$$\frac{dV}{d\theta}(\theta) = \frac{\partial f}{\partial \theta}(x^*(\theta), \theta)$$
The constrained case

Consider a simple maximization problem

$$\max_{x} f(x, \theta) \quad (3)$$

s.t. $g(x, \theta) \leq 0$

where $x$ is a choice variable and $\theta$ is a parameter that we do not control. We assume that $f$ is a concave function of $x$ and that $g$ is a convex function of $x$ so that the first order conditions are not only necessary but sufficient for maximization.
The Lagrangian

The Lagrangian is the following:

\[ L(x, \theta) = f(x, \theta) - \lambda g(x, \theta) \]  

An interior solution of (3) where the constraint is binding satisfies the following first order conditions:

\[ \frac{\partial L}{\partial x}(x, \theta) = 0 \]

\[ \frac{\partial L}{\partial \theta}(x, \theta) = 0. \]
First Order Conditions

Equivalently

\[ \frac{\partial f}{\partial x}(x, \theta) = \lambda \frac{\partial g}{\partial x}(x, \theta) \]  \hspace{1cm} (5)

\[ g(x, \theta) = 0 \]  \hspace{1cm} (6)

The solution to this equation is the maximizer we are looking for. Let’s denote it by \( x^*(\theta) \). It is clearly a function of \( \theta \).
Implications

Note that by plugging $x^*(\theta)$ into the constraint function $g$, we get the following identity (see (6)):

$$g(x^*(\theta), \theta) \equiv 0.$$ 

Taking derivatives we get

$$\frac{\partial g}{\partial x}(x^*(\theta), \theta) \frac{\partial x^*}{\partial \theta}(\theta) + \frac{\partial g}{\partial \theta}(x^*(\theta), \theta) = 0$$

or

$$\frac{\partial g}{\partial x}(x^*(\theta), \theta) \frac{\partial x^*}{\partial \theta}(\theta) = -\frac{\partial g}{\partial \theta}(x^*(\theta), \theta) \quad \text{(7)}$$
Let's plug this optimal value into the objective function and define the optimal value of the objective function as a function of $\theta$.

\[
V(\theta) \equiv f(x^*(\theta), \theta).
\]

We are interested in what happens to the optimal value of $f$ when the parameter $\theta$ changes. Formally, we are interested in

\[
\frac{dV}{d\theta}(\theta).
\]

Note that $V$ will change both because $\theta$ affects $f$ and because it also affect the optimal choice of $x$. 

The answer

To answer this question we take derivatives:

\[
\frac{dV}{d\theta}(\theta) = \frac{\partial f}{\partial \theta}(x^*(\theta), \theta) + \frac{\partial f}{\partial x}(x^*(\theta), \theta) \frac{\partial x^*}{\partial \theta}(\theta) + \lambda \frac{\partial g}{\partial x}(x^*(\theta), \theta) \frac{\partial x^*}{\partial \theta}(\theta)
\]

\[
= \frac{\partial f}{\partial \theta}(x^*(\theta), \theta) + \lambda \frac{\partial g}{\partial x}(x^*(\theta), \theta) \frac{\partial x^*}{\partial \theta}(\theta) - \lambda \frac{\partial g}{\partial \theta}(x^*(\theta), \theta)
\]

\[
= \frac{\partial f}{\partial \theta}(x^*(\theta), \theta) - \lambda \frac{\partial g}{\partial \theta}(x^*(\theta), \theta) + \frac{\partial L}{\partial \theta}(x^*(\theta), \theta)
\]

where the second equality follows from (5), the third one from (7), and the last one from (4).
This proves the envelope theorem: the total rate of change in the optimal value of the objective function due to a small change in the parameter $\theta$ is simply the rate of change in the Lagrangian $L$ evaluated at the optimal value of $x$.

\[
\frac{dV}{d\theta}(\theta) = \frac{\partial L}{\partial \theta}(x^*(\theta), \theta)
\]