Solving 1st-order & 2nd-order differential equations under constant inputs

1st-order:
\[ \dot{x} + \frac{x}{\tau} = f \]  (\( \tau \): time constant, \( f \): constant input)

Solution: \( x(t) = k_0 + k_1 e^{\frac{t}{\tau}} \),
where \( \alpha_1 \) is a solution of \( \dot{x} + \frac{x}{\tau} = 0 \) \( \Rightarrow \alpha_1 = -\frac{1}{\tau} \)

To find \( k_0 \) and \( k_1 \), use \( x(\infty) \) and \( x(0) \):
\[ \begin{cases} x(\infty) = k_0 \\ x(0) = k_0 + k_1 \end{cases} \]

2nd-order:
\[ \ddot{x} + 2\zeta \omega_0 \dot{x} + \omega_0^2 x = f \]  (\( \zeta \): damping ratio, \( \omega_0 \): undamped freq., \( f \): cont. input)

Solution: \( x(t) = \begin{cases} k_0 + k_1 e^{\alpha_1 t} + k_2 e^{\alpha_2 t} & \text{if } \zeta \neq 1 \\ k_0 + k_1 e^{\alpha_1 t} + k_2 e^{\alpha_2 t} & \text{if } \zeta = 1 \end{cases} \),
where \( \alpha_1, \alpha_2 \) solutions of \( \dot{\alpha}^2 + 2\zeta \omega_0 \dot{\alpha} + \omega_0^2 = 0 \) \( \Rightarrow \alpha_{1,2} = \omega_0 \left[ -\zeta \pm \sqrt{\zeta^2 - 1} \right] \).

Note when \( \zeta = 1 \), \( \alpha_1 = \alpha_2 = -\omega_0 \).

To find \( k_0, k_1, k_2 \), use \( x(\infty), \dot{x}(0), \ddot{x}(0) \):
\[ \begin{cases} x(\infty) = k_0 \\ \dot{x}(0) = \{ k_0 + k_1 + k_2 & \zeta \neq 1 \\ k_0 + k_1 & \zeta = 1 \} \\ \ddot{x}(0) = \{ k_1 \alpha_1 + k_2 \alpha_2 & \zeta \neq 1 \\ k_1 \alpha_1 + k_2 \alpha_2 & \zeta = 1 \} \end{cases} \]

Inductor: \( v(t) = L \frac{di}{dt} \), \( w(t) = \frac{1}{2} L I^2(t) \), \( L = \frac{\mu N^2 A}{l} \)

Capacitor: \( \dot{x}(t) = C \frac{dv}{dt} \), \( w(t) = \frac{1}{2} C V^2(t) \), \( C = \frac{\varepsilon A}{d} \)

Note: \( k_0 \) is constant only when input \( f \) is constant.
For example, \( f = Ae^{bt} \Rightarrow k_0 = Ce^{bt} \)
$t = 0$  

**RC** Circuit  

$t < 0$: $V_C(t) = V_c(0)$  
(Capacitor voltage remains constant since no current flows for $t < 0$)  

$t > 0$:  

$$V = i(t)R + V_c(t) = \left(\frac{c}{R} \frac{dV_C}{dt}\right) R + V_c(t)$$  

$$\Rightarrow \frac{dV_C}{dt} + \frac{V_c(t)}{RC} = \frac{V}{RC}$$  

$$\Rightarrow V_C(t) = k_0 + k_1 e^{-\frac{t}{RC}}$$  

$t = 0 \Rightarrow V_C(t) = V_C(0) = k_0 + k_1 \Rightarrow k_0 = V$  
$t = 0 \Rightarrow V_C(t) = V = k_0$  
$k_1 = V_C(0) - V$  
(capacitor acts as open ckt. in long run (steady state))  

Thus  

$$V_C(t) = V + (V_C(0) - V) e^{-\frac{t}{RC}}$$

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**Graphs**  

- $V_C(t)$ vs $t$: Charging  
- $V_C(t)$ vs $t$: Discharging

**Time-constant** = time to charge/discharge at constant initial rate.

Initial rate:  

$$\left.\frac{dV_C(t)}{dt}\right|_{t=0} = \left[V_C(0) - V\right] e^{-\frac{t}{RC}} \left(-\frac{1}{RC}\right) \bigg|_{t=0} = -\frac{[V - V_C(0)]}{RC}$$

Time to change at initial const. rate:  

$$\frac{V - V_C(0)}{\left(\frac{V - V_C(0)}{RC}\right)} = RC$$