Transfer function

Impulse response, \( h(t) \leftrightarrow \text{Transfer function } H(s) \)

\[ y(t) = h(t) \ast x(t) \quad \leftrightarrow \quad Y(s) = H(s) \ast X(s). \]

For differential equation system model:

\[ a_N y^{(N)}(t) + \cdots + a_0 y(t) = b_M z^{(M)}(t) + \cdots + b_0 z(t) \]

Since \( e^{st} \) produces output \( H(s) e^{st} \), (see eigen-function derivation)

\[ a_N \frac{d^N}{dt^N} [e^{st} H(s)] + \cdots + a_0 [e^{st} H(s)] = b_M \frac{d^M}{dt^M} [e^{st}] + \cdots + b_0 [e^{st}] \]

\[ \Rightarrow a_N s^N e^{st} H(s) + \cdots + a_0 e^{st} H(s) = b_M s^M e^{st} + \cdots + b_0 e^{st} \]

\[ \Rightarrow [a_N s^N + \cdots + a_0] H(s) = [b_M s^M + \cdots + b_0] \]

\[ \Rightarrow H(s) = \frac{b_M s^M + \cdots + b_0}{a_N s^N + \cdots + a_0} \]

Example:

\[ \frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 2 y(t) = 2 \frac{d}{dt} z(t) - 3 z(t) \]

\[ \Rightarrow H(s) = \frac{2s - 3}{s^2 + 3s + 2} \]

Example (DC Motor):

Torque, \( T = K_1 \dot{x}(t) \), Back EMF, \( V(t) = K_2 \frac{d\theta}{dt} \)

\[ J \frac{d^2 \theta}{dt^2} = K_1 \left[ \frac{z(t)}{\ell} - K_2 \frac{d\theta}{dt} \right] \]

\[ \Rightarrow J \frac{d^2 \theta}{dt^2} + \frac{k_1 k_2}{R} \frac{d\theta}{dt} = \frac{k_1}{R} z(t) \]

\[ \Rightarrow H(s) = \frac{K_1}{J s^2 + \frac{k_1 k_2}{R}} \]
Relationship between Causality/Stability & Impulse Response

\[ \text{Stable} \iff \int_0^{\infty} |h(t)| \, dt < \infty \quad \text{Causal} \iff h(t) = 0 \text{ for } t < 0 \]

Suppose system is stable

i) \( H(s) \) has pole in \( \text{lhp} \) \( \Rightarrow \)
    impulse response decaying to the right
    So for system to be stable, \( h(t) \) must be right-sided

Suppose system is causal \( \Rightarrow \) \( h(t) \) is right-sided

i) \( H(s) \) has pole in \( \text{lhp} \) \( \Rightarrow \)
    impulse response decaying to the right \( \Rightarrow \) stable.

ii) \( H(s) \) has pole in \( \text{rhp} \) \( \Rightarrow \)
    impulse response growing to right \( \Rightarrow \) unstable.

Example: \( H(s) = \frac{2}{s+3} + \frac{1}{s-2} \)

Suppose system is stable.

- \( s = -3 \) in \( \text{lhp} \) \( \Rightarrow \) for stability must be right-sided
- \( \frac{2}{s+3} \iff 2e^{-3t} u(t) \)

- \( s = 2 \) in \( \text{rhp} \) \( \Rightarrow \) for stability must be left-sided
- \( \frac{1}{s-2} \iff -e^{2t} u(-t) \).

Suppose system is causal

\( h(t) \) must be right-sided. So \( h(t) = 2e^{-3t} u(t) + 2e^{2t} u(t) \).

Note: \( H(s) \) cannot be both stable and causal.
Inverse System

\[ H(s) H(s) = 1 \]

\[ H_{inv}(s) = \frac{1}{H(s)} \]

Zeros of \( H(s) \) = poles of \( H_{inv}(s) \), and vice-versa.

For \( H_{inv}(s) \) to be stable & causal
- poles of \( H_{inv}(s) \) in lhp
- zeros of \( H(s) \) in lhp

System with both zeros & poles in lhp is called \textbf{minimum phase}.
A non-minimum phase system has zeros in rhp and cannot have stable inverse.

\textbf{Minimum phase systems}: Unique relationship between magnitude & phase response
**Matlab Commands**

- \( H(s) = \frac{4s^2+6}{s^3+s^2-6} \)

num = [4 0 6];

den = [1 1 0 -6];
y = tf(num, den)

- \( H(s) = \frac{2(s+3)}{(s-1)^2(s+4)} \)

k = 2;

z = -3;

p = [-1, -1, 4];
y = zpk(z, p, k)

- \( A = \begin{bmatrix} 1 & -2 & 3 & 4 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}; \quad D = 0; \quad \gamma = \text{ss}(A, B, C, D).

- [num, den] = ss2tf(A, B, C, D);

- [r, p, k] = zpk2ss(A, B, C, D);

- [num, den] = zpk2tf(A, B, C, D);

- [A, B, C, D] = zpk2ss(z, p, k);

- [r, p, k] = tf2zpk(num, den);

- [A, B, C, D] = tf2ss(num, den);

- roots(num);

- mm = poly(r)

- [r, p, k] = residue(num, den);

- [num, den] = residue(r, p, k);

- [num, den] = residue(num, den);

- z = zeros(sys);

- p = poly(sys);

- ppmap(sys);

- w = 0:100:1000;

- [mag, phase] = bode(sys, w);

- gr = bode(sys);

- k = freqresp(sys, w);

- Hmag = abs(squeeze(H1));

- plot(w, Hmag);

- t = 0:1:30;

- f = exp(zt);

- y = impulse(sys, t);

- plot(t, y);

- y = step(sys, t);

- y = heaviside(sys, f, t);

- sys = t;

- ilaplace(z);

- ilaplace(x);