Root-locus for discrete-time system

- Root-locus can be used to analyze discrete-time system stability.

\[
L(z) = \frac{\frac{1}{2} K z}{(z-1)(z-\frac{1}{2})} \quad \Rightarrow \quad T(z) = \frac{\frac{1}{2} K z}{\frac{1}{2} K z + (z-1)(z-\frac{1}{2})}
\]

- Zeros, N = 1; Poles, N = 2; zero = \{0\}; poles = \{1, \frac{1}{2}\}.
- Branches = \max(N, N) = 2; Start at 1, \frac{1}{2}; end at 0, \infty.

- Asymptotic angle, \(\Theta_k = \frac{(2k+1)\pi}{N-N} = \frac{(2k+1)\pi}{N-N} \int_{0}^{\infty} k = 0 \Rightarrow \Theta_k = \pi\).

- Breakaway points: \(\frac{1}{L(z)} = \frac{(z-1)(z-\frac{1}{2})}{\frac{1}{2} K z} \Rightarrow \frac{d}{dz}(\frac{1}{L(z)}) = \frac{1}{2 K} \left[ \frac{2 \left[ (z-1)(z-\frac{1}{2}) \right] - (z-1)(z-\frac{1}{2})}{z^2} \right]
\]

\[\frac{dL(z)}{dz} = 0 \Rightarrow 2(2z^2 - \frac{3}{2}) - (z^2 - \frac{3}{2} z + \frac{1}{2}) = 0 \]
\[3z^2 - \frac{3}{2} z = 0 \Rightarrow z = \pm \sqrt{\frac{1}{2}} \]

- Imaginary axis crossover point: \(\frac{1}{2} K z + (z-1)(z-\frac{1}{2}) = z^2 + \frac{1}{2} K z + \frac{1}{2}
\]

\[z^2 + 1 = \frac{1}{2}\]
\[z^2 + \frac{1}{2} = 0 \quad \Rightarrow \text{zero becomes zero when } K = 3\]

Aux. polynomial: \(z^2 + \frac{1}{2} = 0 \Rightarrow z = \pm \sqrt{\frac{1}{2}} j\)

Instability occurs when \(z = -1\)
\[\Rightarrow \frac{1}{2} K z + (z-1)(z-\frac{1}{2}) = 0 \Rightarrow -\frac{1}{2} K + (-2) \left( \frac{1}{2} \right) = 0 \]
\[\Rightarrow K = 6\]
Nyquist Stability Criterion

Consider a function \( F(s) \) and a counterclockwise contour \( C \) (a closed curve) such that \( F(s) \) has no poles on \( C \), and \( F(s) \) is "analytic" inside \( C \) except for a finite number of points.

As \( s \) moves along \( C \), \( F(s) \) moves along \( \Gamma \).

\[ \text{# of counterclockwise encirclements of } F(s) \text{ along } \Gamma \text{ due to one counterclockwise encirclement of } s \text{ along } C' = z - p \]

\[ z = \# \text{ of zeros of } F(s) \text{ inside } C; \quad p = \# \text{ of poles of } F(s) \text{ inside } C. \]

We are interested in RHP poles of \( T(s) = \frac{L(s)}{1 + L(s)} \)

\[ = \text{RHP roots of } 1 + L(s); \quad 1 + L(s) = 0 \implies L(s) = -1 \]

Curve \( C \) encircling RHP and bypassing any imaginary-axis poles of \( 1 + L(s) \)

(= poles of \( L(s) \)).

\[ \text{# RHP zeros of } 1 + L(s) - \# \text{ RHP poles of } 1 + L(s) = \# \text{ counterclockwise encirclement of } (-1,0) \text{ by } L(s) \]

\[ = \# \text{ RHP poles of } T(s) = \# \text{ counterclockwise encirclement of } (-1,0) \text{ by } L(s) \]

\[ + \# \text{ RHP poles of } L(s) \]

\[ \# \text{ imaginary poles of } T(s) = \# \text{ times } L(s) \text{ passes } (-1,0). \]
Nyquist Stability criterion

Example: \[ L(s) = \frac{6}{(s+1)(s+2)(s+3)} \]

\( \Rightarrow \) on imaginary axis, \[ L(j\omega) = \frac{6}{(j\omega+1)(j\omega+2)(j\omega+3)} \]

\[ |L(j\omega)| = \frac{6}{(\omega^2+1)^{\frac{1}{2}} (\omega^2+4)^{\frac{1}{2}} (\omega^2+9)^{\frac{1}{2}}} \]

\[ \angle L(j\omega) = -\tan^{-1}(\omega) - \tan^{-1}(\frac{\omega}{2}) - \tan^{-1}(\frac{\omega}{3}) \]

\# RHP \( L(s) = 0 \); \# counter-clockwise encirclement of \((-1,0)\) by Nyquist curve = 0

\( \Rightarrow \) \# RHP poles of \( T(s) = 0 \)

Also, Nyquist curve does not pass through \((-1,0)\) \( \Rightarrow \) No imag. axis poles of \( T(s) \).

Stability Margins:

\( \omega_p \): phase crossover freq. (freq. where phase = \( \pi \))

\( \omega_g \): gain crossover freq. (freq. where gain = 1)

Gain margin = \[ \left( \frac{1}{|L(j\omega)|} \right) \text{ in dB} = 20\log_{10}\left( \frac{1}{4} \right) = -20\log_{10} a \]

Phase margin = \( \left( \frac{|L(j\omega)|}{\omega = \omega_g} \right) - \pi = \theta \)

Max sensitivity = \[ \frac{1}{\min(1+|L(j\omega)|)} = \frac{1}{\eta} \]

Nyquist curve can be drawn for discrete-time systems by choosing the curve \( C \) along perimeter of unit circle.

Math commands: roots, rlocus(sys), rlocfind(sys), nyquist(sys), margin(sys)

Here \( sys \) is loop gain \( L(s) \).