Chapter 9: Filter & Equalizer Design

- Both filter & equalizer shape signals in freq. domain.
  - Filters filter-out unwanted freq. components
  - Equalizers compensate for distortion introduced by dispersive medium.

**Review of notation:**

**FT:** \( \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = X(j\omega) \quad \Rightarrow \quad X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \)

**DFT:** \( \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j2\pi n k/N} = X[k] \quad \Rightarrow \quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \)

- **Ideal Low Pass Filters:** To get an understanding of filters, we first consider Ideal filters, and later real filters. Also, transforms can be used to convert a low-pass filter to any other filter (band-pass, high-pass, or band-stop). So attention is given to low-pass filters.

Let us first consider an "all-pass" or "distortionless" system:

\[ x(t) \rightarrow \text{distortionless} \rightarrow y(t) = x(t-t_0) \]

\[ (H(j\omega)) \rightarrow H(j\omega) = e^{-j\omega t_0} \]

So an ILPF must have:

\[ H(j\omega) = \begin{cases} e^{-j\omega t_0} & |\omega| \leq \omega_c \\ 0 & |\omega| > \omega_c \end{cases} \]

Then, \( h_{\text{ILPF}}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\omega t_0} e^{j\omega t} d\omega = \frac{1}{2\pi} \left[ \frac{e^{-j\omega(t-t_0)}}{-\omega} \right]_{\omega=-\infty}^{\omega=\infty} = \frac{\sin (\omega_c (t-t_0))}{\pi (t-t_0)} \]

This is noncausal! A real filter must be causal.
Design of real filters

- Need to approximate ideal filter by a causal and stable system (see the figure for specification of approximation).
- Need to realize approximating system using physical means.

There is no unique way to approximate an ideal filter, and various schemes have been proposed based on criteria such as:
- Maximally flat: An order-M filter is maximally flat if the degree of denominator in transfer function
  \[ \frac{d^k}{dw^k} \left| H(iw) \right| = 0 \text{ for } k = 0, \ldots, 2M-1 \]
- Equiripple response: In pass-band, \( |H(iw)| \) oscillates between 1 and \( 1 - \epsilon \), i.e., rippling is "uniform."

Butterworth filter of order M:

Note: \( |H(iw)|^2 \mid_{w=0} = 1 \)

\[ |H(iw)|^2 \mid_{w=w_c} = \frac{1}{2} \Rightarrow w_c \text{ is cut-off freq. (gain is } \frac{1}{2} \text{ of max.)} \]

\[ |H(iw)|^2 \mid_{w=w_p} = (1-\epsilon)^2 \Rightarrow w_p = w_c \left( \frac{\epsilon}{1-\epsilon} \right)^{1/2M} \]

\[ |H(iw)|^2 \mid_{w=w_s} = 8 \Rightarrow w_s = w_c \left( \frac{1-\epsilon}{8} \right)^{1/2M} \]

Near \( w=0 \), \( |H(iw)|^2 = 1 + \frac{1}{2} \left( \frac{w}{w_c} \right)^{2M} + \frac{3}{8} \left( \frac{w}{w_c} \right)^{4M} + \frac{5}{16} \left( \frac{w}{w_c} \right)^{6M} + \ldots \)

\( \Rightarrow \) Maximally flat.

(Cube Taylor Series)
Design of real filters (cont.)

**Design of Butterworth filter:** Given magnitude freq. response, how to go about designing the transfer function that is stable & causal?

\[ |H(j\omega)|^2 = H(j\omega)H^*(j\omega) = H(j\omega)H(-j\omega) = H(s)H(-s) \mid_{s=j\omega}. \]

So setting \( s = 2\pi f \), we get

\[ H(s)H(-s) = \frac{1}{1 \pm \left( \frac{2\pi f}{\omega_c} \right)^2}. \]

The poles of \( H(s)H(-s) \) are located at \( s = j\omega_c(-1)^{1/2M} \),

\[ = \omega_c e^{j\pi \left[ \frac{1}{2} + (2k+1)/2M \right]}, \quad k = 0, \pm 1, \pm 2, \ldots \]

Thus, the poles are located uniformly on circle of radius \( \omega_c \) (see figure for \( M = 3 \)).

Half the poles are for \( H(s) \), and the other half for \( H(-s) \).

For \( H(s) \) to be stable and causal, poles must be in left half plane.

For \( M = 3 \), \( H(s) \) has poles at

\[ s = \omega_c, \quad s = \omega_c e^{j\pi/3}, \quad s = \omega_c e^{-j\pi/3}, \quad s = \omega_c(-1/2+j\sqrt{3}/2), \quad \text{and} \quad \omega_c(1/2-j\sqrt{3}/2). \]

So, \( H(s) = \frac{1}{(\omega_c/s + 1)(\omega_c/s + \frac{1}{2} - j\sqrt{3}/2)(\omega_c/s + \frac{1}{2} + j\sqrt{3}/2)} \)

\[ = \frac{1}{\left( \frac{\omega_c}{s} \right)^3 + 2 \left( \frac{\omega_c}{s} \right)^2 + 2 \left( \frac{\omega_c}{s} \right) + 1}. \]

**Table 8.1** lists normalized \((\omega_c = 1)\) filters.

For \( \omega_c \neq 1 \), replace \( s \) by \( \frac{s}{\omega_c} \) in filter transfer function.

**Examples:** Find \( H(s) \) for \( M = 1, M = 2 \).
Chebyshev/Inverse Chebyshev/Elliptic filters

- Butterworth filters maintain flatness near DC, but don't have a sharp transition between pass and stop bands. Chebyshev filters provide sharper transitions at the expense of rippling in pass/stop bands. The ripples are uniform.

- Inverse Chebyshev filters have no ripples in pass band, but have ripples in stop band.

- Elliptic filters have ripples in both pass and stop bands. Elliptic filters offer sharpest transitions between the pass and stop bands for any fixed order. Transfer functions for inverse Chebyshev filters contain infinite zeros in s-plane.

Filter Transformations

- Normalized LPF $\rightarrow$ General LPF $\beta \rightarrow \frac{\beta}{\omega_c}$ (cut-off freq)

- Normalized LPF $\rightarrow$ General HPF $\beta \rightarrow \frac{\omega_c}{\beta}$ (center freq)

- Normalized LPF $\rightarrow$ General BPF $\beta \rightarrow \frac{\beta^2 - \omega_d^2}{\beta\omega_b}$ (Bandwidth)

- Normalized LPF $\rightarrow$ General BLP $\beta \rightarrow \frac{\beta\omega_b}{\beta^2 - \omega_d^2}$

Passive filters (passive $\Rightarrow$ no power supply used; gain)

Active filters (uses op-amps) and digital filters are more popular

Exercise: Analyze ckt TF.