EE 324 LAB 5
Approximating Continuous-time Systems with Discrete-time Systems

In this lab, you will learn how to derive a discrete-time system as an approximation of a continuous-time system.

We consider two popular methods for transforming a continuous-time system into discrete-time systems which approximate the behavior of the continuous-time system. In other words, we want to derive transformations from $H(s)$ to $H(z)$. Let $T$ denote the sampling time.

Prelab:
Problem 1: The Euler Approximation
The mathematical basis behind this transformation is to approximate the derivative as follows:

$$\frac{dx}{dt}(kT) \approx \frac{x[(k+1)T] - x(kT)}{T}$$

or, using $k$ instead of $kT$ for convenience,

$$\dot{x}(k) \approx \frac{1}{T}[x(k + 1) - x(k)]$$

and applying this approximation to the differential equation.

1. From the transfer function $H(s)$ (in Laplace domain) of the system, derive the differential equation that characterizes it.

2. Evaluate the equation at time $kT$.

3. Use Euler’s approximation to derive a difference equation, typical of a discrete system.

4. From the difference equation, compute the transfer function $H_E(z)$ of the resulting discrete system (in $Z$ domain).

5. What are the relations between the poles and the zeros of the continuous-time system, and the poles and the zeros of the discrete-time system, respectively? Can the discrete system be unstable? Why?

6. Verify that $H_E(z)$ can be directly obtained from $H(s)$ by taking the transformation $s = \frac{z-1}{T}$.
Problem 2: The Trapezoidal Approximation

This transformation is based on approximating the integral with the trapezoidal rule. Recall that the integral:

\[ x(t) = \int_0^T \dot{x}(\tau) d\tau, \]

can be written as:

\[ x[(k + 1)T] = x(kT) + \int_{kT}^{(k+1)T} \dot{x}(\tau) d\tau \cong x(kT) + \frac{[\dot{x}((k + 1)T) + \dot{x}(kT)] T}{2}, \]

where the area described by the integral is approximated by the area of a trapezoid with height \( T \) and the two bases \( \dot{x}((k + 1)T) \) and \( \dot{x}(kT) \) respectively. Using \( k \) instead of \( kT \) for convenience,

\[ \frac{\dot{x}(k + 1) + \dot{x}(k)}{2} \cong \frac{x(k + 1) - x(k)}{T} \]

which amounts to a more precise, but implicit approximation of the derivative.

1. Evaluate the differential equation derived previously at times \( kT \) and \( (k + 1)T \).

2. Use the trapezoidal approximation to derive a difference equation, typical of a discrete system.

3. From the difference equation, compute the transfer function \( H_T(z) \) of the resulting discrete system (in \( Z \)-domain).

4. What are the relations between the poles and the zeros of the continuous-time system and the poles and zeros of the discrete-time system, respectively? Can the discrete system be unstable? Why?

5. Verify that \( H_T(z) \) can be directly obtained from \( H(s) \) by taking the transformation \( s = \frac{2z - 1}{Tz + 1} \).

Laboratory Assignment:

Use Simulink to simulate the continuous-time system, as well as the discrete-time systems from the two approximation methods for different sampling times \( T \) and different inputs, discuss your findings, and suggest a \( T \) which provides a satisfactory approximation for the two methods.