Section 8.3
Representing Relations

Connection Matrices

Let $R$ be a relation from

$$A = \{a_1, a_2, \ldots, a_m\}$$

to

$$B = \{b_1, b_2, \ldots, b_n\}.$$  

**Definition:** An $m \times n$ connection matrix $M$ for $R$ is defined by

$$M_{ij} = 1 \text{ if } <a_i, b_j> \text{ is in } R,$$

$$= 0 \text{ otherwise.}$$

Example:

We assume the rows are labeled with the elements of $A$ and the columns are labeled with the elements of $B.$

Let

$$A = \{a, b, c\}$$

$$B = \{e, f, g, h\}$$

$$R = \{<a,e>, <c, g>\}$$
Then the connection matrix $M$ for $R$ is

$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
$$

Note: the order of the elements of $A$ and $B$ matters.

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**Theorem:** Let $R$ be a binary relation on a set $A$ and let $M$ be its connection matrix. Then

- $R$ is reflexive iff $M_{ii} = 1$ for all $i$.
- $R$ is symmetric iff $M$ is a symmetric matrix: $M = M^T$
- $R$ is antisymmetric if $M_{ij} = 0$ or $M_{ji} = 0$ for all $i \neq j$.

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**Combining Connection Matrices**

**Definition:** the *join* of two matrices $M_1$, $M_2$, denoted $M_1 \lor M_2$, is the component wise boolean ‘or’ of the two matrices.

**Fact:** If $M_1$ is the connection matrix for $R_1$ and $M_2$ is the connection matrix for $R_2$ then the join of $M_1$ and $M_2$, $M_1 \lor M_2$, is the connection matrix for $R_1 \cup R_2$. 

Definition: the meet of two matrices $M_1, M_2$, denoted $M_1 \land M_2$ is the componentwise boolean ‘and’ of the two matrices.

Fact: If $M_1$ is the connection matrix for $R_1$ and $M_2$ is the connection matrix for $R_2$ then the meet of $M_1$ and $M_2$, $M_1 \land M_2$ is the connection matrix for $R_1 \cap R_2$.

Obvious questions:

Given the connection matrix for two relations, how does one find the connection matrix for

- The complement?
- The relative complement?
- The symmetric difference?

The Composition

Definition: Let

$M_1$ be the connection matrix for $R_1$ and

$M_2$ be the connection matrix for $R_2$. 
The boolean product of two connection matrices $M_1$ and $M_2$, denoted $M_1 \otimes M_2$, is the connection matrix for the composition of $R_2$ with $R_1$, $R_2 \circ R_1$.

$$(M_1 \otimes M_2)_{ij} = \bigvee_{k=1}^{n} [(M_1)_{ik} \wedge (M_2)_{kj}]$$

Why?

In order for there to be an arc $<x, z>$ in the composition then there must be and arc $<x, y>$ in $R_1$ and an arc $<y, z>$ in $R_2$ for some $y$!

The Boolean product checkes all possible $y$’s. If at least one such path exists, that is sufficient.

Note: the matrices $M_1$ and $M_2$ must be conformable: the number of columns of $M_1$ must equal the number of rows of $M_2$.

If $M_1$ is mxn and $M_2$ is nxp then $M_1 \otimes M_2$ is mxp.
Example:

\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

\[M_1=\begin{bmatrix}
0 & 0 \\
0 & 1 \\
1 & 0 \\
0 & 1 \\
\end{bmatrix}\]

\[M_2=\begin{bmatrix}
0 & 0 \\
0 & 1 \\
1 & 0 \\
0 & 1 \\
\end{bmatrix}\]

\[M_1 \otimes M_2 =\begin{bmatrix}
0 & 1 \\
0 & 1 \\
\end{bmatrix}\]

\[(M_1 \otimes M_2)_{12} = [(M_1)_{11} \land (M_2)_{12}] \lor [(M_1)_{12} \land (M_2)_{22}] \lor [(M_1)_{13} \land (M_2)_{32}] \lor [(M_1)_{14} \land (M_2)_{42}]\]
\[ [0 \land 0] \lor [1 \land 1] \lor [0 \land 0] \lor [0 \land 1] = 1 \]

Note:

- there is an arc in \( R_1 \) from node 1 in A to node 2 in B
- there is an arc in \( R_2 \) from node 2 in B to node 2 in C.
- Hence there is an arc in \( R_2 \circ R_1 \) from node 1 in A to node 2 in C.

A useful result:

\[ M_{R^n} = M^n_R \]

Digraphs

(see section 8.1)

Given the digraphs for \( R_1 \) and \( R_2 \), find the digraphs for

- \( R_2 \cup R_1 \)
- \( R_2 \cap R_1 \)
- \( R_2 - R_1 \)
\[ R_2 \oplus R_1 \]
\[ \overline{R}_1 \]