Section 1.3
Predicates and Quantifiers

A generalization of propositions - *propositional functions* or *predicates*.: propositions which contain variables

Predicates become propositions once every variable is *bound* by

- assigning it a value from the *Universe of Discourse* $U$

or

- quantifying it

Examples:

Let $U = \mathbb{Z}$, the integers $= \{ \ldots -2, -1, 0, 1, 2, 3, \ldots \}$

- $P(x): x > 0$ is the predicate. It has no truth value until the variable $x$ is bound.

Examples of propositions where $x$ is assigned a value:

- $P(-3)$ is false,
- $P(0)$ is false,
- $P(3)$ is true.
The collection of integers for which $P(x)$ is true are the positive integers.

• $P(y) \lor \neg P(0)$ is not a proposition. The variable $y$ has not been bound. However, $P(3) \lor \neg P(0)$ is a proposition which is true.

• Let $R$ be the three-variable predicate $R(x, y, z): x + y = z$

Find the truth value of

$R(2, -1, 5), \ R(3, 4, 7), \ R(x, 3, z)$
Quantifiers

• Universal

P(x) is true for every x in the universe of discourse.

Notation: universal quantifier

\[ \forall x P(x) \]

‘For all x, P(x)’, ‘For every x, P(x)’

The variable x is bound by the universal quantifier producing a proposition.

Example: U={1,2,3}

\[ \forall x P(x) \iff P(1) \land P(2) \land P(3) \]

• Existential

P(x) is true for some x in the universe of discourse.

Notation: existential quantifier

\[ \exists x P(x) \]
‘There is an $x$ such that $P(x)$,’ ‘For some $x$, $P(x)$’, ‘For at least one $x$, $P(x)$’, ‘I can find an $x$ such that $P(x)$.’

Example: $U = \{1,2,3\}$
\[ \exists x P(x) \iff P(1) \lor P(2) \lor P(3) \]

• Unique Existential

$P(x)$ is true for one and only one $x$ in the universe of discourse.

Notation: unique existential quantifier

\[ \exists! x P(x) \]

‘There is a unique $x$ such that $P(x)$,’ ‘There is one and only one $x$ such that $P(x)$,’ ‘One can find only one $x$ such that $P(x)$.’
Example:

$U = \{1, 2, 3\}$

**Truth Table:**

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<th>P(1)</th>
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<th>$\exists ! x P(x)$</th>
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How many minterms are in the DNF?

Note:

**REMEMBER!**

A predicate is not a proposition until *all* variables have been bound either by quantification or assignment of a value!

Equivalences involving the negation operator

$\neg \forall x P(x) \leftrightarrow \exists x \neg P(x)$

$\neg \exists x P(x) \leftrightarrow \forall x \neg P(x)$
Distributing a negation operator across a quantifier changes a universal to an existential and vice versa.

Multiple Quantifiers: read left to right . . .

Example: Let $U = \mathbb{R}$, the real numbers, 

$P(x,y): xy= 0$

\[
\forall x \forall y P(x, y) \\
\forall x \exists y P(x, y) \\
\exists x \forall y P(x, y) \\
\exists x \exists y P(x, y)
\]

The only one that is false is the first one.

Suppose $P(x,y)$ is the predicate $x/y=1$?

Example:

Let $U = \{1,2,3\}$. Find an expression equivalent to

\[
\forall x \exists y P(x, y)
\]

where the variables are bound by substitution instead:
Expand from inside out or outside in.

Outside in:

\[ \exists y P(1, y) \land \exists y P(2, y) \land \exists y P(3, y) \]
\[ \iff [P(1,1) \lor P(1,2) \lor P(1,3)] \land 
  [P(2,1) \lor P(2,2) \lor P(2,3)] \land 
  [P(3,1) \lor P(3,2) \lor P(3,3)] \]

Converting from English

(can be very difficult)

Examples:

\( F(x): x \text{ is a fleegle} \)
\( S(x): x \text{ is a snurd} \)
\( T(x): x \text{ is a thingamabob} \)

\( U = \{ \text{fleegles, snurds, thingamabobs} \} \)

(Note: the equivalent form using the existential quantifier is also given)

- Everything is a fleegle

\[ \forall x F(x) \]
\[ \iff \neg \exists x \neg F(x) \]
• Nothing is a snurd.

\[ \forall x \neg S(x) \]
\[ \iff \neg \exists x S(x) \]

• All fleegles are snurds.

\[ \forall x [F(x) \rightarrow S(x)] \]
\[ \iff \forall x [\neg F(x) \lor S(x)] \]
\[ \iff \forall x \neg [F(x) \land \neg S(x)] \]
\[ \iff \neg \exists x [F(x) \land \neg S(x)] \]

• Some fleegles are thingamabobs.

\[ \exists x [F(x) \land T(x)] \]
\[ \iff \neg \forall x [\neg F(x) \lor \neg T(x)] \]

• No snurd is a thingamabob.

\[ \forall x [S(x) \rightarrow \neg T(x)] \]
\[ \iff \neg \exists x [S(x) \land T(x)] \]

• If any fleegle is a snurd then it's also a thingamabob

\[ \forall x [(F(x) \land S(x)) \rightarrow T(x)] \]
\[ \iff \neg \exists x [F(x) \land S(x) \land \neg T(x)] \]
Extra Definitions:

• An assertion involving predicates is valid if it is true for every universe of discourse.

• An assertion involving predicates is satisfiable if there is a universe and an interpretation for which the assertion is true. Else it is unsatisfiable.

• The scope of a quantifier is the part of an assertion in which variables are bound by the quantifier

Examples:

Valid: $\forall x \neg S(x) \leftrightarrow \neg \exists x S(x)$

Not valid but satisfiable: $\forall x [F(x) \rightarrow T(x)]$

Not satisfiable: $\forall x [F(x) \land \neg F(x)]$

Scope: $\forall x [F(x) \lor S(x)]$ vs. $\forall x [F(x)] \lor \forall x [S(x)]$

Dangerous situations:

• Commutativity of quantifiers
\( \forall x \forall y P(x, y) \iff \forall y \forall x P(x, y) \)?

YES!

\( \forall x \exists y P(x, y) \iff \exists y \forall x P(x, y) \)?

NO!

DIFFERENT MEANING!

- Distributivity of quantifiers over operators

\( \forall x [P(x) \land Q(x)] \iff \forall x P(x) \land \forall x Q(x) \)?

YES!

\( \forall x [P(x) \rightarrow Q(x)] \iff [\forall x P(x) \rightarrow \forall x Q(x)] \)?

NO!

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