Online Sparse + Low-Rank Matrix Recovery

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(joint work with Wei Lu, Chenlu Qiu and Brian Lois)

Acknowledgements

- This talk is based on joint work with my students
  - Wei Lu and Jinchun Zhan (online sparse matrix recovery – Modified-CS)
  - Chenlu Qiu and Brian Lois (online sparse + low-rank matrix recovery / robust PCA)
- Funded by NSF grants CCF-1117125 and CCF-0917015
- Other collaborators: Han Guo (new student) and Prof. Leslie Hogben (Math, ISU)

Recovery from incomplete data: the question

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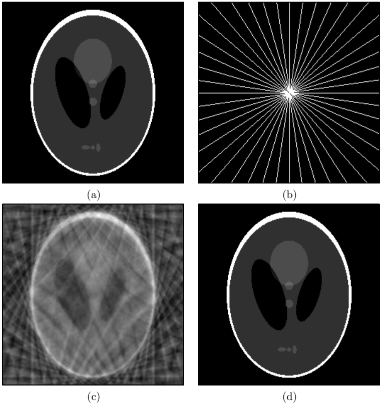
$$\text{image} = \text{background} + \text{foreground}$$

  - Question: can we recover two image sequences from one?
  - Yes: if exploit the low-rank structure of the background sequence and sparseness of the foreground

Sparse recovery: Magnetic Resonance Imaging (MRI)



Example taken from [Candes,Romberg,Tao,T-IT, Feb 2006]

- (a) Shepp-Logan phantom: 256×256 image
- (b) MR imaging pattern: 256-point DFT along 22 radial lines
- (c) Inverse-DFT
- (d) Basis Pursuit solution (uses sparsity: gives exact recovery!)

Sparse recovery / Compressive sensing [Mallat et al'93], [Feng,Bresler'96], [Gordinsky,Rao'97], [Chen,Donoho'98], [Candes,Romberg,Tao'05],[Donoho'05]

- Recover a sparse vector x, with support size at most s, from

$$y := Ax + w$$

when A is a known fat matrix and $\|w\|_2 \leq \epsilon$ (small noise).

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- Solution by convex relaxation: ℓ_1 minimization [Chen,Donoho'98]:

$$\min \|\tilde{x}\|_1 \text{ subject to } \|y - A\tilde{x}\|_2 \leq \epsilon$$

if $\delta_{2s}(A) < 0.4$, error bounded by $C\epsilon$ [Candes et al'05,'06,'08]

  - restricted isometry constant (RIC) $\delta_s(A)$: smallest real # s.t.

$$(1-\delta_s)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1+\delta_s)\|x\|_2^2$$

  for all s-sparse vectors x [Candes,Tao,T-IT'05]

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- Applications: projection imaging - MRI, CT, astronomy, single-pixel camera

Low-rank matrix recovery (completion)

- Recover a low-rank matrix from a subset of its entries

$$Y := \mathcal{P}_\Omega(L)$$

Ω is the set of missing entries [Fazel et al, Recht et al,2009]

- Applications: recommendation system design, e.g. Netflix problem; survey data analysis, ...
  - ℓ_k: ratings of movies by user k
  - a given user will rate only a subset of all the movies: missing entries; goal: complete the matrix in order to recommend movies
  - matrix is low-rank: user preferences governed by only a few factors

Sparse + Low-rank matrix recovery

- Separate a low-rank matrix L and a sparse matrix X from

$$Y := X + L$$

  or from a subset of entries of $(X + L)$
  - if L or range(L) is the quantity of interest: robust PCA
  - if X is quantity of interest: robust sparse recovery

- Applications: video analytics (e.g. for surveillance, tracking, mobile video chat, occlusion removal,...) [Candes et al,2009]

$$X = [x_1, x_2 \ldots, x_t, \ldots x_{t_{\max}}], \; L = [\ell_1, \ell_2, \ldots \ell_t, \ldots \ell_{t_{\max}}]$$

  - ℓ_t: bg - usually slow changing, global (dense) changes
  - x_t: fg - sparse, consists of one or more moving objects (technically x_t: (fg-bg) on fg support)

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- Other apps: detecting anomalous connectivity patterns in social networks or in computer networks; functional MRI based brain activity detection; recommendation system design

Our work: the question

- How to solve the above problems for dynamically arriving data?
  - e.g., dynamic or functional MRI, online video analytics, ...
- Option 1: batch methods
  - recover the entire sequence in a batch fashion (e.g. for sparse recovery - use Fourier sparsity along the time axis)
  - slow and memory-intensive

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- Option 2: do not use past knowledge
  - fast and memory-efficient, but will need more measurements
- Option 3: design recursive algorithms (our work)
  - use previously recovered images and current observed data to recover the current image
  - fast and memory-efficient and need fewer measurements

Our work: Online (recursive) solutions

- Developed provably accurate recursive solutions for
  - "online" sparse matrix recovery (recursive recovery of sparse signal sequences) [KF-CS, ICIP'08]
    - brief overview
  - "online" sparse + low-rank matrix recovery (online or recursive robust PCA) [Qiu,Vaswani,Allerton 2010]
    - most of this talk
- **The "online" problem as we define it uses extra assumptions**
- **In this talk "recursive" $\Leftrightarrow$ "online" (used interchangeably)**

Recursive recovery of sparse seq's: Problem [Vaswani,ICIP'08][1]

- Given measurements
$$y_t := Ax_t + w_t, \quad \|w_t\|_2 \le \epsilon, \quad t = 0, 1, 2, \ldots$$
  - $A = H\Phi$ (given): $n \times m$, $n < m$
    - H: measurement matrix, Φ: sparsity basis matrix
    - e.g., in MRI: H = partial Fourier, Φ = inverse wavelet
  - y_t: measurements (given)
  - x_t: sparsity basis vector
  - $\mathcal{N}_t$: support set of x_t
  - w_t: small noise
- Goal: recursively reconstruct x_t from $y_0, y_1, \ldots y_t$,
  - i.e. use only y_t and $\hat{x}_{t-1}$ for recovering x_t

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- Goal: recursively reconstruct x_t from $y_0, y_1, \ldots y_t$,
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- Use slow support change: $|\mathcal{N}_t \setminus \mathcal{N}_{t-1}| \approx |\mathcal{N}_{t-1} \setminus \mathcal{N}_t| \ll |\mathcal{N}_t|$
  - also use slow signal value change when valid

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Recursive recovery of sparse seq's: Solutions [KF-CS, ICIP'08], [LS-CS,T-SP,Aug10]

- Introduced Kalman filtered CS (KF-CS) and Least Squares CS (LS-CS):
  - first recursive algorithms that needed fewer measurements for accurate recovery than simple ℓ_1
  - able to obtain time-invariant error bounds on LS-CS error under weaker RIP assumptions (fewer meas's) than simple ℓ_1
- But these could not achieve *exact* recovery with fewer meas's than what simple ℓ_1 needed
  - solved by Modified-CS

Recursive recovery of sparse seq's: Modified-CS [Modified-CS,ISIT'09,T-SP'10,T-IT'15]

- Idea: support at $t-1$, $\mathcal{N}_{t-1}$, is a good predictor of $\mathcal{N}_t$
- Reformulate: Sparse Recovery with Partial Support Knowledge $\mathcal{T}$
  - $\text{support}(x) = \mathcal{T} \cup \Delta \setminus \Delta_e$: Δ, Δ_e unknown

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- Modified-CS: tries to find a vector $\tilde{x}$ that is sparsest outside $\mathcal{T}$ among all vectors satisfying the data constraint

$$\min_{\tilde{x}} \|\tilde{x}_{\mathcal{T}^c}\|_1 \text{ subject to } \|y - A\tilde{x}\|_2 \leq \epsilon$$

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- Provably exact recovery in noise-free case if $\delta_{s+|\Delta|+|\Delta_e|} < 0.4$ [Vaswani,Lu, ISIT'09,T-SP'10]

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- For noisy case: time-invariant error bounds under a realistic signal change model and $\delta_{s+ks_a} < 0.4$ [Zhan,Vaswani, ISIT'13, T-IT'15 (to appear)]
- Regularized modified-CS & modified-CS-residual: also use slow signal value change (when valid);
  - significant advantage over existing work for dynamic MRI

Online Robust PCA: background

- Principal Components' Analysis (PCA): estimate the low-dimensional subspace that best approximates a given dataset
  - SVD on data matrix, compute top left singular vectors
- Robust PCA: PCA in presence of outliers; many useful heuristics in older work, e.g., RSL [De la Torre et al,2003]
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- [Candes et al,2009] posed robust PCA as: separate low-rank matrix L, sparse X from

$$Y := X + L$$

A practical provably correct solution: PCP

- [Candes et al,2009; Chandrasekharan et al,2009; Hsu et al,2011] introduced and studied a convex opt program called PCP:

$$\min_{\tilde{X},\tilde{L}} \|\tilde{L}\|_* + \lambda\|\tilde{X}\|_1 \text{ s.t. } Y = \tilde{X} + \tilde{L}$$

- If (a) left and right singular vectors of L are dense enough; (b) support of X is generated uniformly at random; (c) rank and sparsity are bounded, then PCP exactly recovers X and L from $Y := X + L$ w.h.p. [Candes et al,2009]
  - [Chandrasekharan et al,2009; Hsu et al,2011]: similar flavor; replace 'unif rand support' by upper bound on # of nonzeros in any row of X.
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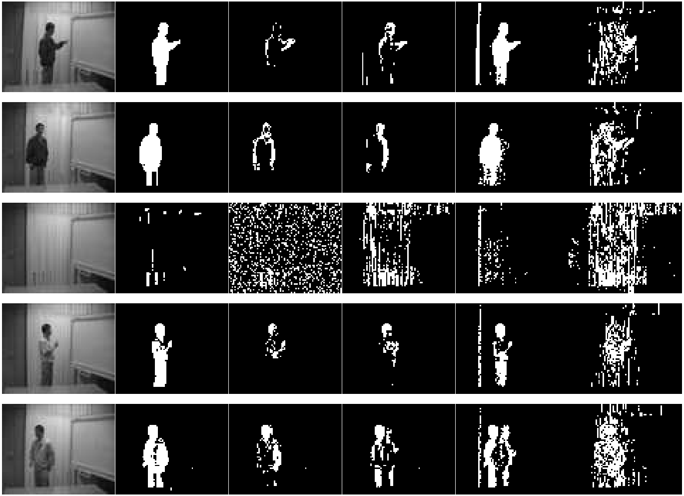
  - first set of guarantees for a practical robust PCA approach
- Much later work on the *batch* robust PCA problem w/ guarantees

Need for an online method

- Disadvantages of batch methods:
  - slower especially for online applications;
  - memory intensive;
  - do not allow infrequent/slow support change of columns of X
    - reason: this can result in X being rank deficient
- Video analytics: need online solution; and have occasionally static or slow moving fg objects
- Functional MRI: the activated brain region does not change a lot from frame to frame
- Network anomaly detection: need online solution; anomalous behavior continues for a period of time after begins

Figure : ReProCS: proposed. Frames $t = t_0 + 60, 120, 199, 475, 1148$.



original | ReProCS (online) | PCP (batch) | RSL (batch) | GRASTA (online) | MG (batch)

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"Online" sparse + low-rank recovery / robust PCA problem

[Qiu,Vaswani,Allerton'10,'11] [Guo,Qiu,Vaswani,T-SP'14] [2]

- Given sequentially arriving n-length data vectors y_t satisfying

$$y_t := \ell_t, \quad t = 1, 2, \ldots, t_0$$

and

$$y_t := x_t + \ell_t, \quad t = t_0 + 1, t_0 + 2, \ldots, t_{\max}$$

  - x_t's are sparse vectors with support sets, $\mathcal{T}_t$, of size at most s;
  - its support sets $\mathcal{T}_t$ have *at least some* changes over time

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  - x_t's are sparse vectors with support sets, $\mathcal{T}_t$, of size at most s;
  - its support sets $\mathcal{T}_t$ have *at least some* changes over time
  - ℓ_t's lie in a slowly-changing low-dimensional subspace of $\mathbb{R}^n$;
    - $\Leftrightarrow \ell_t = P_t a_t$ w/ $\|(I - P_{t-1}P_{t-1}{}')\ell_t\|_2 \ll \|\ell_t\|_2$ (P_t: tall)

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- Goal: recursively estimate x_t, ℓ_t and range(L_t) at all $t > t_0$.

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[Qiu,Vaswani,Allerton'10,'11] [Guo,Qiu,Vaswani,T-SP'14] [3]

- Initial outlier-free seq $y_t = \ell_t$ for first t_0 frames needed to estimate the initial subspace P_{t_0}. Easy to obtain in many apps, e.g.,
  - in video surveillance, easy to get a short background-only training sequence before fg objects start appearing
  - for fMRI, this corresponds to acquiring a short sequence without any activation

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  - for fMRI, this corresponds to acquiring a short sequence without any activation
- Note: extension of all our ideas to the undersampled case $y_t = Ax_t + B\ell_t$ is easy (relevant to MRI apps)

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Related work

Batch robust PCA and performance guarantees

- Older work, e.g. RSL [de la Torre et al,IJCV'03]; PCP and much later work on provably correct robust PCA solutions

Recursive / incremental / online robust PCA algorithms

- Older work (before PCP): [Li et al, ICIP 2003] iRSL: doesn't work
- [Qiu,Vaswani, Allerton'10, Allerton'11, T-SP'14]: ReProCS (Recursive Projected CS)
- [Balzano et al, CVPR 2012]: GRASTA
- [Mateos et al, JSTSP 2013]: batch, online; online: not enough info, no code

Online robust PCA performance guarantees: almost no work

- [Qiu,Vaswani,Lois,Hogben, ISIT'13, T-IT'14]: partial result;
- [Lois, Vaswani,ICASSP'15,arXiv:1409.3959]: complete correctness result
- [Feng et al,NIPS'13 OR-PCA Stoch Opt]: partial result and only asymptotic

Some definitions

- P is a basis matrix $\Leftrightarrow P'P = I$
- "Estimate P" $\Leftrightarrow$ estimate range(P): subspace spanned by col's of P
- "$\hat{P}$ is an accurate estimate of P" $\Leftrightarrow$ $\text{SE}(\hat{P}, P) := \|(I - \hat{P}\hat{P}')P\|_2 \ll 1$

ReProCS algorithm [Qiu,Vaswani,Allerton'10,Allerton'11],[Guo,Qiu,Vaswani,T-SP'14][4]

Recall: for $t > t_0$, $y_t := x_t + \ell_t$, $\ell_t = P_t a_t$, P_t: tall $n \times r$ basis matrix

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- Projection: compute $\tilde{y}_t := \Phi_t y_t$, where $\Phi_t := I - \hat{P}_{t-1}\hat{P}'_{t-1}$
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- Subspace update: update $\hat{P}_t$ every α frames by projection-PCA

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Why ReProCS works [Qiu,Vaswani,Lois,Hogben,T-IT,2014] [5]

- Slow subspace change: noise β_t seen by sparse recovery step is small
- Denseness of columns of $P_t \Rightarrow$ RIC of $\Phi_t = I - \hat{P}_{t-1}\hat{P}'_{t-1}$ is small
  - denseness assump: $(2s)\max_t \max_i \|(P_{t-1})_{i,:}\|_2^2 \le 0.09$
  - easy to show [Qiu,Vaswani,Lois,Hogben,T-IT,2014]:

$$\delta_{2s}(\Phi_t) = \max_{|T|\le 2s} \|I_T{}'\hat{P}_{t-1}\|_2^2 \le (2s)\max_i \|(\hat{P}_{t-1})_{i,:}\|_2^2 \le 0.09{+}0.05$$

(here: 0.05 is due to the small error b/w $\hat{P}_{t-1}$ and P_{t-1})

- Above two facts + any result for ℓ_1 min: x_t is accurately recovered; and hence $\ell_t = y_t - x_t$ is accurately recovered

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- Above two facts + any result for ℓ_1 min: x_t is accurately recovered; and hence $\ell_t = y_t - x_t$ is accurately recovered
- Most of the work: show accurate subspace recovery $\hat{P}_t \approx P_t$
  - std PCA results not applicable: $e_t := \ell_t - \hat{\ell}_t = x_t - \hat{x}_t$ correlated w/ ℓ_t

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ReProCS algorithm: why projection-PCA needed

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ReProCS correctness result [Lois,Vaswani, arXiV:1409.3959],[Qiu,Vaswani,Lois,Hogben,T-IT'14][6]

For most videos (i.e. w.p. at least $1 - n^{-10}$),

- the region occupied by the foreground objects (support of x_t) is exactly recovered at all times, and

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- background changes (w.r.t. a mean background image) are dense,
- there is some motion of the foreground objects at least once every so often (there is some change in the support of x_t's)

Details follow in the next few slides . . .

ReProCS correctness result: Support change - examples

1. *(random motion)* all support sets mutually disjoint
   - this satisfies our model as long as $s \in O(\frac{n}{\log n})$

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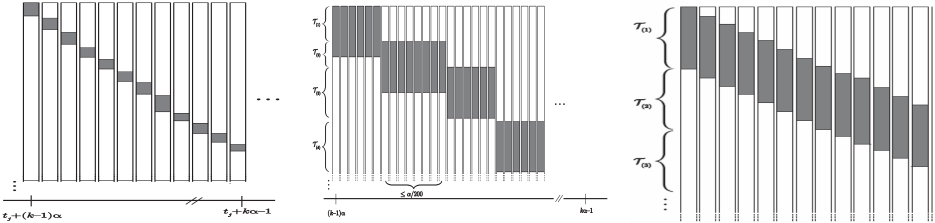
3. *(slow motion)* an object of length s moves down by at least one pixel in every frame
   - this satisfies our model as long as $s \in O(\log n)$

ReProCS correctness result: Support change - examples



(a) disjoint supports (b) infrequent motion (c) slow moving

Figure : In any of these we could have randomly selected pixels (need not be a block) at a given time and also random ordering across time

ReProCS correctness result: Subspace change model

ℓ_t's are zero mean, bounded and mutually independent r.v.'s with covariance matrix Σ_t that is low-rank and "slowly changing"

- $\Sigma_t \overset{EVD}{=} P_t \Lambda_t P_t'$ where $P_t = P_j$ for $t \in [t_j, t_{j+1} - 1]$, $j = 1, 2, \ldots J$
- P_j is a tall $n \times r_j$ basis matrix that changes as

$$P_j = [P_{j-1} \setminus P_{j,\text{old}}, \ \ P_{j,\text{new}}]$$

- "slow change": $\lambda_{\text{new}}^+(d) := \max_{t \in [t_j, t_j + d]} \lambda_{\max}(\Lambda_{t,\text{new}})$ is small and $t_{j+1} - t_j$ is large

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Define

- $c := \max_j \text{rank}(P_{(j),\text{new}})$, $\gamma_{\text{new}}(d) := \max_{t \in [t_j, t_j + d]} \|a_{t,\text{new}}\|_\infty$
- $r := r_0 + Jc$, $\lambda^+ := \max_t \lambda_{\max}(\Lambda_t)$, $\gamma := \max_t \|a_t\|_\infty$

Theorem

Consider ReProCS. Pick a $\zeta \leq \min\left(\frac{10^{-4}\lambda_0^-}{(r_0+Jc)^2\lambda^+}, \frac{1}{(r_0+Jc)^3\gamma^2}\right)$. *If ReProCS algorithm parameters* α, K, ξ, ω *are set appropriately, and if*

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1. *initial subspace accurately estimated:* $\|(I - \hat{P}_0\hat{P}_0')P_0\|_2 \leq r_0\zeta$
2. *"slow subspace change" holds:*
   - *projection of* ℓ_t *along new direc's small for first* d *frames after* t_j*: for a* $d \geq (K+2)\alpha$, $\lambda_{new}^+(d) \leq 3\lambda_0^-$ *and* $\gamma_{new}(d) \leq 0.05 x_{\min}$
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3. *subspace basis matrices are dense enough:*

$$(2s)\max_i \|(P_{j,new})_{i,:}\|_2^2 \leq 0.0004 \text{ and } (2s)\max_i \|(P_J)_{i,:}\|_2 \leq 0.09$$

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4. *support of* x_t *has size smaller than* s *and changes enough,*
   - *e.g., moves down by at least* $s/10$ *pixels at least once every* $\alpha/500$ *frames,*

then, with probability at least $1 - n^{-10}$,

1. *support*(x_t) *is exactly recovered at all times,*
2. $SE_t := \|(I - \hat{P}_t \hat{P}_t')P_t\|_2$ *reduces to* $(r + c)\zeta$ *within* $(K + 2)\alpha$ *frames after* t_j,
3. $\|\ell_t - \hat{\ell}_t\|_2 = \|x_t - \hat{x}_t\|_2 \leq b \ll \|x_t\|_2$

Notice: no bound needed on λ^+ *or on* γ*: the result allows large but structured* ℓ_t

Details:

- *B. Lois and N. Vaswani, A Correctness Result for Online Robust PCA, ICASSP 2015, arXiV:1409.3959.*
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Discussion: Contributions

- To our knowledge, first correctness result for online robust PCA
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- New proof techniques needed: useful for various other problems
  - almost all existing robust PCA results are for batch approaches
  - previous PCA results require $e_t := \hat{\ell}_t - \ell_t$ uncorrelated w/ ℓ_t

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- Needs knowledge of bounds on γ_{new} and c to set algorithm parameters
- Needs a tighter bound on rank and sparsity compared to PCP
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- Only ensures accurate recovery of x_t, ℓ_t, not exact

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- Easy extension to $y_t = Ax_t + B\ell_t$
- Relax independence assumption on ℓ_t's, replace by AR model (ongoing) – almost exactly same result
- Result for ReProCS-deletion – ReProCS that also deletes direc's (ongoing):
  - needs an extra clustering assumption on the eigenvalues for a certain period of time after subspace change has stabilized;
  - but relaxes denseness requirement and so allows $r_{\text{mat}} \in O(n)$ instead of $r_{\text{mat}} \in O(\log n)$

Online Matrix Completion

- Can provide a provably accurate solution for online matrix completion; that also allows highly correlated set of unknown entries
  - but requires slow subspace change and initial subspace knowledge
- Low-rank matrix completion is a special case w/ known $T_t = \text{support}(x_t)$
  - in MC: T_t is the set of unknown entries of ℓ_t at time t
- ReProCS for online matrix completion:
  - Assume: accurate initial subspace knowledge, $\hat{P}_0$.
  - Compute $\Phi_t := (I - \hat{P}_{t-1}\hat{P}_{t-1}')$
  - Given T_t, get an estimate of ℓ_t as

$$\hat{\ell}_t = (I - I_{T_t}(\Phi_t)_{T_t}{}^{\dagger}\Phi_t)y_t$$

  - Use projection-PCA as before to update the subspace estimate

ReProCS algorithm - recap [Qiu,Vaswani,Allerton'10,Allerton'11][7]

Initialize: given $\hat{P}_0$ with range$(\hat{P}_0) \approx$ range$([\ell_1, \ell_2, \dots \ell_{t_0}])$
For $t > t_0$,

- Projection: compute $\tilde{y}_t := \Phi_t y_t$, where $\Phi_t := I - \hat{P}_{t-1}\hat{P}_{t-1}'$
  - then $\tilde{y}_t = \Phi_t x_t + \beta_t$, $\beta_t := \Phi_t \ell_t$ is small "noise"
- Noisy Sparse Recovery: ℓ_1 min + support estimate + LS: get $\hat{x}_t$
  - $\hat{x}_{t,cs} = \arg\min_x \|x\|_1$ s.t. $\|\tilde{y}_t - \Phi_t x\|_2 \leq \xi$
  - $\hat{\mathcal{T}}_t = \{i : |(\hat{x}_{t,cs})_i| > \omega\}$
  - $\hat{x}_t = I_{\hat{\mathcal{T}}_t}({A_{\hat{\mathcal{T}}_t}}' A_{\hat{\mathcal{T}}_t})^{-1} {A_{\hat{\mathcal{T}}_t}}' y_t$
- Get $\hat{\ell}_t = y_t - \hat{x}_t$
- Subspace update: update $\hat{P}_t$ every α frames by projection-PCA

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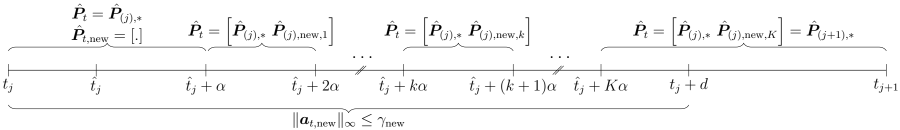
[7] C. Qiu and N. Vaswani, Real-time Robust Principal Components' Pursuit, Allerton, 2010
C. Qiu and N. Vaswani, Recursive Sparse Recovery in Large but Correlated Noise, Allerton 2011

ReProCS algorithm: projection PCA

Assume $t_{j+1} - t_j > (K+2)\alpha$; recall: t_j: subspace change times



let $\hat{P}_{j,*} := \hat{P}_{j-1}$ be an (accurate) estimate of the previous subspace

at $t = \hat{t}_j + k\alpha,\ k = 1, 2, \dots K$,

- $\hat{P}_{j,\text{new},k} \leftarrow SVD\left((I - \hat{P}_{j,*}\hat{P}'_{j,*})[\hat{\ell}_{\hat{t}_j+(k-1)\alpha+1}, \dots \hat{\ell}_{\hat{t}_j+k\alpha}], thresh\right)$
- update $\hat{P}_t = [\hat{P}_{j,*}, \hat{P}_{j,\text{new},k}]$

Proof idea: Why projection PCA works?

- Before the first proj-PCA, i.e. for $t \in [t_j, \hat{t}_j + \alpha]$,
  - $P_t = [P_*, P_{\text{new}}]$, $\hat{P}_{t-1} = [\hat{P}_*] \Rightarrow \beta_t$ (noise seen by sparse rec step) and hence $e_t = \hat{x}_t - x_t = \ell_t - \hat{\ell}_t$ is largest
  - e_t still not too large due to slow subspace change; and e_t is sparse and supported on $\mathcal{T}_t$
  - at $t = \hat{t}_j + \alpha$, get $\hat{P}_{\text{new},1}$: estimate is good because of above: $\text{SE}(P_{\text{new}}, \hat{P}_{\text{new},1}) := \|(I - \hat{P}_{\text{new},1}\hat{P}_{\text{new},1}{}')P_{\text{new}}\|_2 < 0.6$

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- For $t \in [\hat{t}_j + \alpha + 1, \hat{t}_j + 2\alpha]$,
  - $P_t = [P_*, P_{\text{new}}]$, $\hat{P}_{t-1} = [\hat{P}_*, \hat{P}_{\text{new},1}] \Rightarrow \beta_t$ and hence e_t smaller; and e_t is sparse and supported on $\mathcal{T}_t$
  - at $t = \hat{t}_j + 2\alpha$, get $\hat{P}_{\text{new},2}$; estimate better because of above
- Continuing this way, show $\text{SE}(P_{\text{new}}, \hat{P}_{\text{new},k}) < 0.6^k + 0.4c\zeta$; pick K so $\text{SE}(P_{\text{new}}, \hat{P}_{\text{new},K}) < c\zeta$

Proof Outline: k-th projection-PCA interval

Conditioned on accurate recovery so far,

- slow subspace change, denseness assumption, appropriate support threshold and LS ensure that $e_t := x_t - \hat{x}_t = \hat{\ell}_t - \ell_t$ satisfies

$$e_t = I_{\mathcal{T}_t}[\Phi_{\mathcal{T}_t}{}'\Phi_{\mathcal{T}_t}]^{-1} I_{\mathcal{T}_t}{}'\Phi \ell_t \text{ where } \Phi := I - \hat{P}_{t-1}\hat{P}_{t-1}{}'$$

  and

$$\|[\Phi_{\mathcal{T}_t}{}'\Phi_{\mathcal{T}_t}]^{-1}\|_2 \leq 1.2$$

- by $\sin\theta$ theorem [Davis,Kahan,1970],

$$\text{SE}(\hat{P}_{\text{new},k}, P_{\text{new}}) \lesssim \frac{\|\text{perturbation}\|_2}{\lambda_{\text{new}}^- - \|\text{perturbation}\|_2}$$

$$\|\text{perturbation}\|_2 \lesssim 2\Big\|\frac{1}{\alpha}\sum_t (I - \hat{P}_*\hat{P}_*{}')\ell_t e_t'\Big\|_2 + \Big\|\frac{1}{\alpha}\sum_t e_t e_t'\Big\|_2$$

- use matrix Hoeffding ineq [Tropp,2012] to bound these terms w.h.p.

Proof Outline: k-th projection-PCA interval – 2

Conditioned on accurate recovery so far,

- the dominant perturbation term

$$\mathrm{dom} := \mathbb{E}\left[\frac{1}{\alpha}\sum_{t=\hat{t}_j+(k-1)\alpha}^{\hat{t}_j+k\alpha}(I-\hat{P}_*\hat{P}_*{}')\ell_t e_t'\right] \approx \frac{1}{\alpha}\sum_t A_t B_t'$$

where $A_t := P_{\text{new}}\Lambda_{t,\text{new}}P_{\text{new}}'$ and $B_t := I_{\mathcal{T}_t}[\Phi_{\mathcal{T}_t}{}'\Phi_{\mathcal{T}_t}]^{-1}I_{\mathcal{T}_t}{}'$

- use slow subspace change to get

$$\|\frac{1}{\alpha}\sum_t A_t A_t'\|_2 \le \max_t \|A_t\|_2^2 \le \lambda_{\text{new}}^+(d)^2 \le 9\lambda_0^{-2}$$

- use model on $\mathcal{T}_t$ to show that

$$\|\frac{1}{\alpha}\sum_t B_t B_t'\|_2 = \|\frac{1}{\alpha}\sum_t I_{\mathcal{T}_t}[\Phi_{\mathcal{T}_t}{}'\Phi_{\mathcal{T}_t}]^{-2}I_{\mathcal{T}_t}{}'\|_2 \le \frac{1}{\alpha}1.2^2\varrho^2\beta \le 0.02$$

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- use Cauchy-Schwartz to get $\|\mathrm{dom}\|_2 \lesssim \sqrt{0.02}\cdot 3\lambda_0^-$

Proof Outline: Overall idea

- Define subspace error, $\text{SE}(P, \hat{P}) := \|(I - \hat{P}\hat{P}')P\|_2$.
- Start with $\text{SE}(P_{j-1}, \hat{P}_{j-1}) \leq r_{j-1}\zeta \ll 1$ at $t = t_j - 1$.
  1. First show that $t_j \leq \hat{t}_j \leq t_j + 2\alpha$
  2. Analyze projected sparse recovery for $t \in [\hat{t}_j, \hat{t}_j + \alpha)$
  3. Analyze proj-PCA at $t = \hat{t}_j + \alpha$: $\text{SE}(P_{j,\text{new}}, \hat{P}_{j,\text{new},1}) \leq 0.6$
  4. Repeat for each of the K projection-PCA intervals: show that
  $$\text{SE}(P_{j,\text{new}}, \hat{P}_{j,\text{new},k}) \leq 0.6^k + 0.4c\zeta$$
  5. Pick K s.t. $0.6^K + 0.4c\zeta \leq c\zeta$. Set $\hat{P}_j = [\hat{P}_{(j-1)}, \hat{P}_{j,\text{new},K}]$
- Thus, at $t = \hat{t}_j + K\alpha - 1$,

  $\text{SE}(P_j, \hat{P}_j) \leq \text{SE}(P_{j-1}, \hat{P}_{j-1}) + \text{SE}(P_{j,\text{new}}, \hat{P}_{j,\text{new},K}) \leq r_{j-1}\zeta + c\zeta = r_j\zeta$
- $t_{j+1} - t_j > (K+2)\alpha$ implies $\text{SE}(P_j, \hat{P}_j) \leq r_j\zeta$ at $t = t_{j+1} - 1$

Experiments [Guo,Qiu,Vaswani,TSP'14][8]

1. Real background simulated foreground: background of moving lake water video with a simulated moving rectangular object overlaid on it; object intensity similar to background intensity and object moving slowly (making it a difficult seq)

2. Real videos

[8] H. Guo, C. Qiu, N. Vaswani, An Online Algorithm for Separating Sparse and Low-Dimensional Signal Sequences From Their Sum", IEEE Trans. SP, Aug 2014

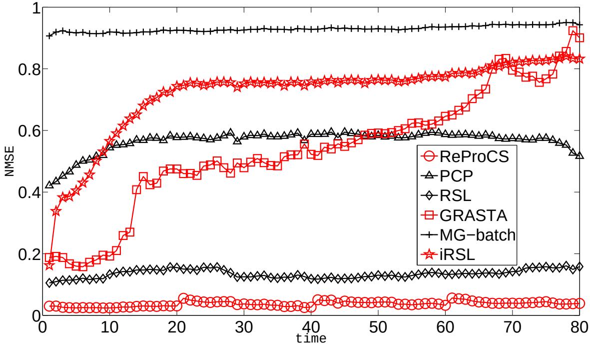

Figure : Recovery error (Monte Carlo over 100 realiz's). Black: batch methods, Red: online methods, Red Circles: ReProCS

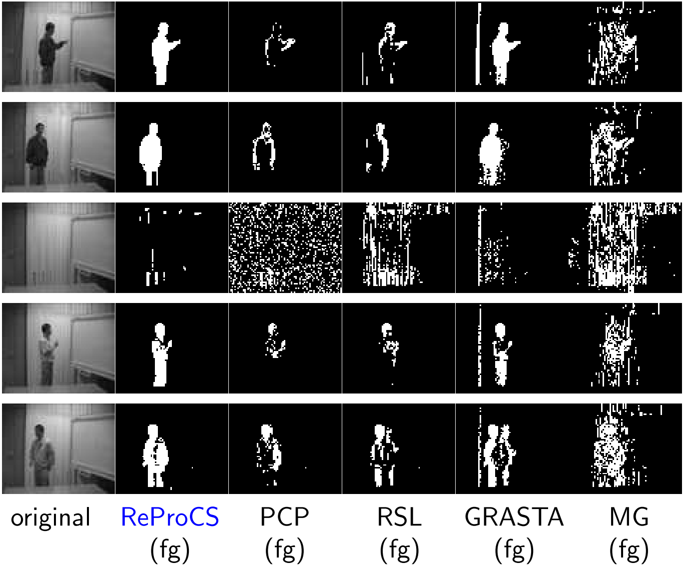

Figure : Online: ReProCS (proposed method) and GRASTA, Batch:

original | ReProCS (bg) | PCP (bg) | RSL (bg) | GRASTA (bg) | MG (bg)

Figure : Background layer recovery at $t = t_{train} + 60, 120, 199, 475, 1148$.

Algorithm parameters

Recall that $\zeta \leq \min(\frac{10^{-4}}{(r_0+Jc)^2 f}, \frac{1}{(r_0+Jc)^3\gamma_*^2})$.

- $\xi = \sqrt{c}\gamma_{\text{new}} + \sqrt{\zeta}(\sqrt{r_0 + Jc} + \sqrt{c})$;
- ω satisfies $7\xi \leq \omega \leq x_{\min} - 7\xi$;
- $K = \left\lceil \frac{\log(0.16c\zeta)}{\log(0.4)} \right\rceil$;
- $\alpha = C(\log(6KJ) + 11\log(n)), \; C \geq C_{add} := 20^2 \cdot 8 \cdot 96^2 \frac{(1.2\xi)^4}{(c\zeta\lambda^-)^2}$
- If we assume that min and max eigenvalues are seen in the training data, then can estimate λ^-, λ^+, γ_* from training data

Summary

- To the best of our knowledge, this is the first correctness result for online sparse + low-rank recovery
  - equivalently also for online robust PCA / recursive sparse recovery in large but structured noise
- Advantages
  - online algorithm: faster; less storage needed; removes a key limitation of PCP: allows more correlated support change
- New proof techniques needed to obtain our results
  - almost all existing robust PCA results are for batch approaches
  - previous finite sample PCA results are not useful: assume $e_t := \hat{\ell}_t - \ell_t$ is uncorrelated with ℓ_t

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