Level Set Method

① Represent contour as zero level ~~of~~ set of a two dim function $\phi(x,y)$

i.e. $C(s) = \{x,y : \phi(x,y) = 0\}$. Many choice of ϕ for a given contour C.

For e.g. $\phi(x,y)$ = Signed Distance Function, i.e.

$$\phi(x,y) = \begin{cases} -d\left(\begin{smallmatrix} x \\ y \end{smallmatrix}, C\right) & (x,y) \text{ inside } C \\ +d\left(\begin{smallmatrix} x \\ y \end{smallmatrix}, C\right) & (x,y) \text{ outside } C \end{cases}$$

$d\left(\begin{smallmatrix} x \\ y \end{smallmatrix}, C\right)$ = minimum distance of (x,y) to any point on the contour C.

② ϕ is zero level set of the contour

Consider an "evolving" function $\phi(x,y,t)$.

By definition, ϕ maintains the property

$$\phi(C_x(s), C_y(s), t) = 0 \quad \forall t.$$

i.e. $\frac{d\phi}{dt} = 0$.

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x}\frac{dC_x}{dt} + \frac{\partial\phi}{\partial y}\frac{dC_y}{dt} + \frac{\partial\phi}{\partial t} = 0$$

$$\frac{\partial\phi}{\partial t} = -\nabla\phi \cdot \frac{dC}{dt}$$

$\frac{dC}{dt} = -\nabla_C E$ can comes from gradient descent of desired energy functional or

$\frac{dc}{dt}$: may come from some random motion applied to the contour (needed for M.C. methods)

e.g. $\frac{dc}{dt} = \underbrace{B_s}_{\text{smoothing term}} \omega_s$

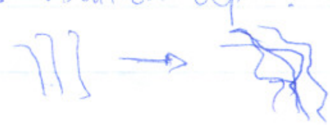
③ Extension velocity : In either case, ~~ae~~ $\frac{dc}{dt}$ is available only on the contour, but we want to ~~apply~~ find a "velocity" for entire ~~contour~~ x-y plane.

Different ways: only need to satisfy smoothness at C.

If perform normal extension (same value along each normal), all level sets ~~move~~ have same motion as zero level set : signed distance function property maintained.

④ Computing zero level set : ~~After Or at any time~~.
After "evolving" the level set function to convergence, need to find the corresponding contour (i.e. zero level set) ~~of the~~

→ use contour command, or bwtrace contour

⑤ Re-initialization : Due to numerical errors, even with normal extension velocities, level sets bunch up : resulting in instability/errors. 

If do not use normal extension : this occurs more often.

(6) Narrowband method

Evolve LSF only in a ± 3 pixel (or other size) narrowband about the zero level set. Re-init whenever zero level set is near boundary of narrowband.

Geometric Repr. of Contour

With parametric repr, Initially chose uniformly spaced points on the contour. Later evolved the point ~~ ~~ according to $\frac{dc}{dt} = -\nabla_c E$.

After sometime, some points will move too close, ~~others~~ more too far away

Also contour may break,

Need to keep re-parameterizing the contour every so often. Re-parameterize: choose points unif on arclength

Also, the contour evolution is defined not ~~ ~~ w.r.t. arclength but ~~w.r.t. some~~ parameter p.

$C_s = \frac{\partial C}{\partial s}$ s is arclength at $t=0$ but not later.

$t > 0$ $ds = \|\frac{\partial C}{\partial p}\| dp = \left[\left(\frac{\partial c_x}{\partial p}\right)^2 + \left(\frac{\partial c_y}{\partial p}\right)^2\right]^{1/2} dp$

$$\frac{\partial c}{\partial s} = \frac{1}{\|\frac{\partial c}{\partial p}\|}\frac{\partial c}{\partial p}$$

Snakes: ignores this distinction.

↓

· Energy is a function of parametric derivative

i.e. $\int \|C_p\|^2 dp = \int_0^{L_0} \|\frac{\partial c}{\partial p}\|^2 dp$

C_p implemented ← $\nabla E = -C_{pp}$.

as $(C_{k+1} - C_k)$ $\int_0^{L_0}$ L_0 = initial contour length.

(assuming unit ~~dist~~ arclength dist)

Ideally:

"geometric energy" $E = \int_0^L ds$. arclength

L = depends on current contour

$\underbrace{\int_{p=0}^{1} \|\frac{\partial c}{\partial p}\| dp.}$ $p: 0$ to 1 or 0 to L_0: same thing: constant integral

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$\boxed{\text{limits are const; easy to differentiate}}$

$$\underbrace{\nabla E = \frac{1}{\|C_p\|}\frac{\partial}{\partial p}\left(\frac{C_p}{\|C_p\|}\right)}$$

w.r.t. $\nabla E \cdot \alpha \triangleq \int (\nabla E)(p)\,\alpha(p)\,\|C_p\|dp$

$\int (\nabla E)(s)\,\alpha(s)\,ds$

"parametric rep"

eg circle: $x = r\cos\theta$, $y = r\sin\theta$ $\quad C(p) = \begin{bmatrix} r\cos 2\pi p \\ r\sin 2\pi p \end{bmatrix}$

$p: 0 \to 1.$

or $C(q) = \begin{bmatrix} r\cos 2\pi q^2 \\ r\sin 2\pi q^2 \end{bmatrix}$: same circle

Implicit: $\Psi(x,y) = x^2+y^2-1 = 0$

$\Psi(x,y) = x^4+y^4+2x^2y^2-1 = 0$

Unit tangent

$$\vec{T} = \frac{\frac{\partial C}{\partial p}}{\left\| \frac{\partial C}{\partial p} \right\|}$$

curvature: $\kappa = \left\| \frac{\partial^2 C}{\partial s^2} \right\| = \frac{1}{\left\| \frac{\partial C}{\partial p} \right\|} \left\| \frac{\partial}{\partial p} \left(\frac{\frac{\partial C}{\partial p}}{\left\| \frac{\partial C}{\partial p} \right\|} \right) \right\|$

$$\frac{\partial^2 C}{\partial s^2} = \kappa \vec{N}$$

"geometric" q: can be expressed in terms of arclength & its derivatives

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can ~~write~~ compute using level set function

e.g. $\vec{N} = \frac{\nabla \phi}{\|\nabla \phi\|}$ (: ϕ is $\perp$ $\vec{T}$ / (ϕ constant along the contour)

$$\kappa = \nabla \cdot \left(\frac{\nabla \phi}{\|\nabla \phi\|} \right)$$