Midterm Exam 2, EE 527, Spring 2008 (Out of 40 points)

1. We have learnt that the Kalman gain, $K_t$, is given by

$$ K_t \triangleq \Sigma_{t\mid t-1} H^T (H \Sigma_{t\mid t-1} H^T + R)^{-1} $$

and the filtering error covariance $\Sigma_{t\mid t} \triangleq [I - K_t H] \Sigma_{t\mid t-1}$.

(a) Show that $K_t$ can also be rewritten as

$$ K_t = (\Sigma_{t\mid t-1}^{-1} + H^T R^{-1} H)^{-1} H^T R^{-1} $$

(b) Use this to show that $\Sigma_{t\mid t}$ can be rewritten as $\Sigma_{t\mid t} = (\Sigma_{t\mid t-1}^{-1} + H^T R^{-1} H)^{-1}$.

8 + 2 points
2. Consider a state space model of the form

\[ Y_t = H_t X_t + r(Y_1, Y_2, \ldots Y_{t-1}) + V_t, \quad V_t \sim N(0, R) \]

\[ X_{t+1} = F_t X_t + q(Y_1, Y_2, \ldots Y_t) + GU_t, \quad U_t \sim N(0, Q) \]

with \( X_0 \sim N(0, \Sigma_0) \). \( X_0, \{U_t, t = 0, \ldots \infty\}, \{V_t, t = 0, \ldots \infty\} \) are mutually independent.

(a) The above is a Kalman model, but with a nonzero “feedback control” input in both equations. Derive the Kalman recursion for it, both prediction and update steps.

(b) Now consider the following nonlinear state space model

\[ Y_t = h(X_t) + V_t, \quad V_t \sim N(0, R) \]

\[ X_{t+1} = f(X_t) + GU_t, \quad U_t \sim N(0, Q) \]

\( X_0 \sim N(0, \Sigma_0) \). \( X_0, \{U_t, t = 0, \ldots \infty\}, \{V_t, t = 0, \ldots \infty\} \) are mutually independent.

Linearize \( h(X_t) \) about \( \hat{X}_{t|t-1} \) and linearize \( f(X_t) \) about \( \hat{X}_{t|t} \). Assume the linearized model to be the true one and apply the Kalman filter derived above. This gives the extended Kalman filter for the nonlinear model.

\[ 5 + 5 = 10 \text{ points} \]
3. Consider the Kalman model but with $U_t$, $V_t$ correlated for the same $t$, i.e.

\[
Y_t = HX_t + V_t, \quad V_t \sim \mathcal{N}(0, R)
\]
\[
X_{t+1} = FX_t + GU_t, \quad U_t \sim \mathcal{N}(0, Q)
\]

where $X_0 \sim \mathcal{N}(0, \Sigma_0)$ and $X_0, \{U_t, t = 0, \ldots, \infty\}$ are mutually independent; $\{V_t, t = 0, \ldots, \infty\}$ are mutually independent; and $U_t$ is independent of $V_\tau, \tau = 1, \ldots, t-1$ and of $V_\tau, \tau = t+1, \ldots, \infty$.

But $U_t, V_t$ are correlated with covariance, $E[U_t V_t^T] = C$.

(a) Derive the expression for $\hat{X}_{t+1|t}$ in terms of $\hat{X}_{t|t}, \hat{X}_{t|t-1}, \Sigma_{t|t}$ and the known matrices.

(b) Derive the expression for $\Sigma_{t+1|t}$ in terms of $\Sigma_{t|t}, \Sigma_{t|t-1}$ and the known matrices.

7 + 3 = 10 points
4. You are given a sequence of observations $Y_0, Y_2, \ldots Y_{N-1}$ which satisfy

$$Y_n = X r^n + w_n, \ w_n \sim \mathcal{N}(0, \sigma^2)$$

with $X \sim \mathcal{N}(0, \sigma_X^2)$. $X, \{w_n, n = 0, \ldots N - 1\}$ are mutually independent. $r$ is a constant.

(a) Use the joint Gaussian formula to find the MMSE estimate of $X$ given $Y_0, Y_1, \ldots Y_{N-1}$.

(b) Can you find the MAP estimate?

9 + 1 = 10 points
Formulas

1. Matrix Inversion Identity

\[(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}\]

2. For jointly Gaussian \(X, Y\) with joint PDF

\[
\begin{bmatrix} Y \\ X \end{bmatrix} \sim \mathcal{N}(\begin{bmatrix} \mu_Y \\ \mu_X \end{bmatrix}, \begin{bmatrix} \Sigma_Y & \Sigma_{YX} \\ \Sigma_{XY} & \Sigma_X \end{bmatrix})
\]

\[
E[X|Y] = \mu_X + \Sigma_{XY}\Sigma_Y^{-1}(Y - \mu_Y)
\]

\[
Cov[X|Y] = \Sigma_X - \Sigma_{XY}\Sigma_Y^{-1}\Sigma_X
\]

Note \(Cov[X|Y] \triangleq E[(X - E[X|Y])(X - E[X|Y])^T|Y]\)

3. Kalman filter equations. For the state space model,

\[
Y_t = H_tX_t + V_t, \quad V_t \sim \mathcal{N}(0, R)
\]

\[
X_{t+1} = F_tX_t + G_tU_t, \quad U_t \sim \mathcal{N}(0, Q)
\]

where \(X_0, \{U_t, t = 1, \ldots, \infty\}, \{V_t, t = 1, \ldots, \infty\}\) are mutually independent and \(X_0 \sim \mathcal{N}(0, \Sigma_0)\), we have

\[
K_t = \Sigma_{t|t-1}H_t^T(H_t\Sigma_{t|t-1}H_t^T + R)^{-1}
\]

\[
\hat{X}_{t|t} = \hat{X}_{t|t-1} + K_t(Y_t - H_t\hat{X}_{t|t-1})
\]

\[
\Sigma_{t|t} = [I - K_tH_t]\Sigma_{t|t-1}
\]

\[
\hat{X}_{t+1|t} = F_t\hat{X}_{t|t}
\]

\[
\Sigma_{t+1|t} = F_t\Sigma_{t|t}F_t^T + G_tQG_t^T
\]

with initialization, \(\hat{X}_{0|\infty} = 0, \Sigma_{0|\infty} = \Sigma_0\). Here

\[
\hat{X}_{t|s} \triangleq E[X_t|Y_{1:s}]
\]

\[
\Sigma_{t|s} \triangleq Cov[X_t|Y_{1:s}]
\]