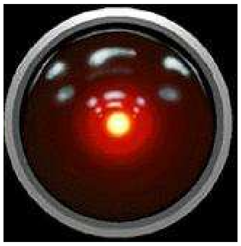


HCI/ComS 575X:
Computational Perception

Instructor: Alexander Stoytchev
http://www.cs.iastate.edu/~alex/classes/2007_Spring_575X/

# The Kalman Filter (part 2)

March 7, 2007



## Project Proposals

- Are due today
- Please post the documents on the Wiki.
- Reviews are due on Sunday March 25-th.

## Final Project Demos

- Wednesday April 25-th
- Two sessions
  - 9:00 -10:30 am
  - 2:30 - 4:00 pm

## Logistics Meeting

- At least one person from each project group should attend.
- When: Wednesday March 21 @ 3pm
- Location: VRAC conference room.

## Equipment for Checkout

Sample HW4 Solutions
(created by the TAs)

Readings for Today's Lecture

Brown and Hwang (1992)

"Introduction to Random Signals and Applied Kalman Filtering"

Ch 5: The Discrete Kalman Filter

Maybeck, Peter S. (1979)

Chapter 1 in ``Stochastic models, estimation, and control'',

Mathematics in Science and Engineering Series, Academic Press.

Arthur Gelb, Joseph Kasper, Raymond Nash, Charles Price, Arthur Sutherland (1974)

Applied Optimal Estimation

MIT Press.

Readings for Next Time
(Monday after Spring Break)

- Particle Filters
- Posted on the class web page

## Let's Start With a Demo

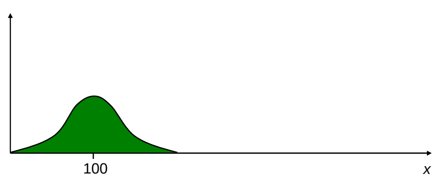
- Matlab Program Written by John Burnett (who took the class last semester)
- Posted on the class web page

## Another Example

[www.cse.lehigh.edu/~spletzer/cse398_Spring05/lec011_Localization2.ppt]

## A Simple Example

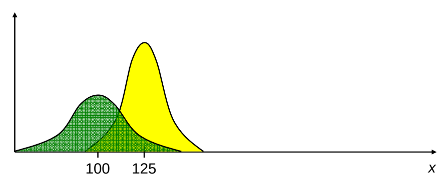
- Consider a ship sailing east with a perfect compass trying to estimate its position.
- You estimate the position *x* from the stars as $z_1=100$ with a precision of $\sigma_x=4$ miles




[www.cse.lehigh.edu/~spletzer/cse398_Spring05/lec011_Localization2.ppt]

## A Simple Example (cont'd)

- Along comes a more experienced navigator, and she takes her own sighting $z_2$
- She estimates the position $x= z_2 =125$ with a precision of $\sigma_x=3$ miles
- How do you merge her estimate with your own?




[www.cse.lehigh.edu/~spletzer/cse398_Spring05/lec011_Localization2.ppt]

## A Simple Example (cont'd)

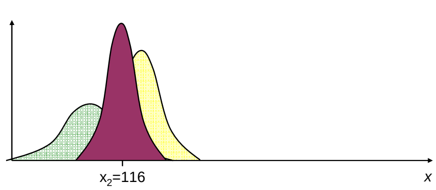
$$\mu = \left[\frac{\sigma_{z_2}^2}{\sigma_{z1}^2+\sigma_{z_2}^2}\right] z_1 + \left[\frac{\sigma_{z_1}^2}{\sigma_{z1}^2+\sigma_{z_2}^2}\right] z_2$$

$$= \left[\frac{9}{16+9}\right]100 + \left[\frac{16}{16+9}\right]125 = 116$$

$$\frac{1}{\sigma^2} = \frac{1}{\sigma_{z_1}^2} + \frac{1}{\sigma_{z_2}^2}$$

$$\frac{1}{\sigma^2} = \frac{1}{9} + \frac{1}{16} = \frac{25}{144}$$

$$\Rightarrow \sigma = 2.4$$




[www.cse.lehigh.edu/~spletzer/cse398_Spring05/lec011_Localization2.ppt]

## A Simple Example (cont'd)

- With the distributions being Gaussian, the best estimate for the state is the mean of the distribution, so…

$$x_2 = \left[\frac{\sigma_{z_2}^2}{\sigma_{z_1}^2+\sigma_{z_2}^2}\right] z_1 + \left[\frac{\sigma_{z_1}^2}{\sigma_{z1}^2+\sigma_{z_2}^2}\right] z_2$$

or alternately

$$= z_1 + \left[\frac{\sigma_{z_1}^2}{\sigma_{z_1}^2+\sigma_{z_2}^2}\right](z_2 - z_1) \quad \text{Correction Term}$$

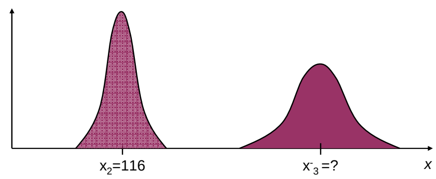
$$= z_1 + K_2(z_2 - z_1)$$

where $K_t$ is referred to as the *Kalman gain*, and must be computed at each time step

[www.cse.lehigh.edu/~spletzer/cse398_Spring05/lec011_Localization2.ppt]

## A Simple Example (cont'd)

- OK, now you fall asleep on your watch. You wake up after 2 hours, and you now have to re-estimate your position
- Let the velocity of the boat be nominally 20 miles/hour, but with a variance of $\sigma^2_w$=4 miles$^2$/hour
- What is the best estimate of your current position?

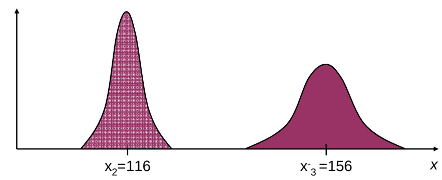



[www.cse.lehigh.edu/~spletzer/cse398_Spring05/lec011_Localization2.ppt]

## A Simple Example (cont'd)

- The next effect is that the gaussian is translated by a distance and the variance of the distribution is increased to account for the uncertainty in dynamics

$$x_3^- = x_2 + v\Delta t \qquad \Rightarrow x_3^- = 116 + 40 = 156$$
$$\sigma_3^{2-} = \sigma_2^2 + \sigma_w^{2-}\Delta t \qquad \Rightarrow \sigma_3^{2-} = 5.76 + 8 = 13.76$$




[www.cse.lehigh.edu/~spletzer/cse398_Spring05/lec011_Localization2.ppt]

## A Simple Example (cont'd)

- OK, this is not a very accurate estimate. So, since you've had your nap you decide to take another measurement and you get $z_3$=*165* miles
- Using the same update procedure as the first update, we obtain

$$x_3 = x_3^- + K_3(z_3 - x_3^-)$$

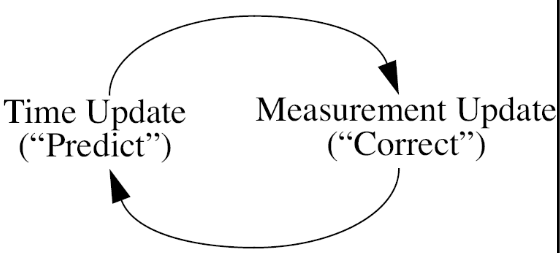
$$\sigma_3^2 = \sigma_3^{2-} - K_3\sigma_3^{2-}$$
$$= 13.76 - \left[\frac{13.76}{13.76+16}\right]13.76 = 7.40$$

and so on…

[www.cse.lehigh.edu/~spletzer/cse398_Spring05/lec011_Localization2.ppt]

## The Predictor-Corrector Approach

- In this example, prediction came from using knowledge of the vehicle dynamics to estimate its change in position
- An analogy with a robot would be integrating information from the robot kinematics (*i.e.* you give it a desired [x, y, α] velocities for a time Δt) to estimate changed in position
- The correction is accomplished through making exteroceptive observations and then fusing this with your current estimate
- This is akin to updating position estimates using landmark information, etc.
- In practice, the prediction rate is typically much higher than the correction




## Calculating the new mean

$$\mu = [\text{Scaling Factor 1}]\,z_1 + [\text{Scaling Factor 2}]\,z_2$$

$$\mu = [\sigma_{z_2}^2/(\sigma_{z_1}^2 + \sigma_{z_2}^2)]z_1 + [\sigma_{z_1}^2/(\sigma_{z_1}^2 + \sigma_{z_2}^2)]z_2$$

## Calculating the new variance

$$\sigma^2 = \boxed{\text{Scaling Factor 1}}\,\sigma_{z_1}^2 + \boxed{\text{Scaling Factor 2}}\,\sigma_{z_2}^2$$

$$[\sigma_{z_2}^2/(\sigma_{z_1}^2+\sigma_{z_2}^2)]^2 \qquad [\sigma_{z_1}^2/(\sigma_{z_1}^2+\sigma_{z_2}^2)]^2$$

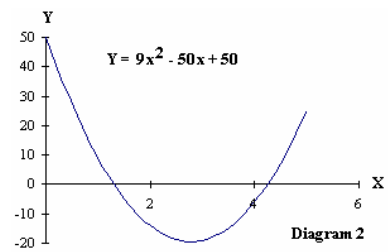
$$\sigma^2 = [\sigma_{z_2}^2/(\sigma_{z_1}^2+\sigma_{z_2}^2)]^2\sigma_{z_1}^2 + [\sigma_{z_1}^2/(\sigma_{z_1}^2+\sigma_{z_2}^2)]^2\sigma_{z_2}^2$$

## What makes these scaling factors special? Are there other ways to combine the two measurements?

- They minimize the error between the prediction and the true value of X.
- They are optimal in the least-squares sense.

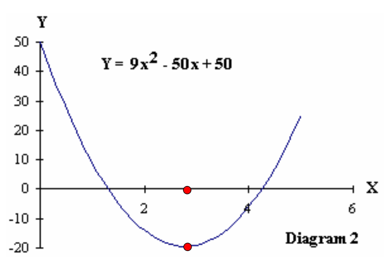
## How can we minimize the error?

## What is the minimum value?




[http://home.ubalt.edu/ntsbarsh/Business-stat/otherapplets/function.gif]

## What is the minimum value?




[http://home.ubalt.edu/ntsbarsh/Business-stat/otherapplets/function.gif]

## Finding the Minimum Value

- $Y = 9x^2 - 50x + 50$
- $dY/dx = 18x - 50 = 0$
- The minimum is obtained when
  x=50/18=2.77777(7)
- The minimum value is
  $Y(x_{min}) = 9*(50/18)^2 - 50*(50/18) + 50 = -19.44444(4)$

## Start with two measurements

$$z_1 = x + v_1 \text{ and } z_2 = x + v_2$$

- $v_1$ and $v_2$ represent zero mean noise

## Formula for the estimation error

- The new estimate is

$$\hat{x} = s_1 z_1 + s_2 z_2$$

- The error is

$$e = \hat{x} - x$$

## Expected value of the error

$$\begin{aligned}
E[e] &= E[\hat{x} - x] \\
&= E[s_1 z_1 + s_2 z_2 - x] \\
&= E[s_1(x + v_1) + s_2(x + v_2) - x] \\
&= s_1 E[x] + s_1 E[v_1] + s_2 E[x] + s_2 E[v_2] - E[x] \\
&= s_1 E[x] + 0 + s_2 E[x] + 0 - E[x] \\
&= s_1 x + s_2 x - x = 0
\end{aligned}$$

- If the estimate is unbiased this should hold

$$\text{Therefore, } s_1 + s_2 - 1 = 0$$

$$\text{which can be rewritten as } s_2 = 1 - s_1$$

## Find the Mean Square Error

$$\begin{aligned}
E[e^2] &= E[(\hat{x} - x)^2] \\
&= ?
\end{aligned}$$

$$\begin{aligned}
E[e^2] &= E[(\hat{x} - x)^2] \\
&= E[\hat{x}^2 - 2\hat{x}x + x^2] \\
&= E[(s_1 z_1 + s_2 z_2)^2 - 2(s_1 z_1 + s_2 z_2)x + x^2] \\
&= E[(s_1(x + v_1) + s_2(x + v_2))^2 - 2(s_1(x + v_1) + s_2(x + v_2))x + x^2] \\
&= E[s_1^2(x + v_1)^2 + 2s_1 s_2(x + v_1)(x + v_2) + s_2^2(x + v_2)^2 - 2s_1(x + v_1)x - 2s_2(x + v_2) \\
&= E[\underline{\underline{s_1^2 x^2}} + \underline{2s_1^2 v_1 x} + s_1^2 v_1^2 + \underline{\underline{2s_1 s_2 x^2}} + \underline{2s_1 s_2 v_1 x} + \underline{2s_1 s_2 v_2 x} + 2s_1 s_2 v_1 v_2 + \\
&\quad + \underline{\underline{s_2^2 x^2}} + \underline{2s_2^2 v_2 x} + s_2^2 v_2^2 - \underline{\underline{2s_1 x^2}} - \underline{2s_1 v_1 x} - \underline{\underline{2s_2 x^2}} - \underline{2s_2 v_2 x} + \underline{\underline{x^2}}] \\
&= E[(s_1^2 + 2s_1 s_2 + s_2^2 - 2s_1 - 2s_2 + 1)x^2 + \\
&\quad +2(s_1^2 v_1 + s_1 s_2 v_1 + s_1 s_2 v_2 + s_2^2 v_2 - s_1 v_1 - s_2 v_2)x + \\
&\quad + s_1^2 v_1^2 + 2s_1 s_2 v_1 v_2 + s_2^2 v_2^2] \\
&= \{(s_1 + s_2)^2 - 2(s_1 + s_2) + 1\} E[x^2] + \\
&\quad +2\{s_1^2 E[v_1] + s_1 s_2 E[v_1] + s_1 s_2 E[v_2] + s_2^2 E[v_2] - s_1 E[v_1] - s_2 E[v_2]\} E[x] + \\
&\quad + s_1^2 E[v_1^2] + 2s_1 s_2 E[v_1]E[v_2] + s_2^2 E[v_2^2] \\
&= (1 - 2 + 1)E[x^2] + 2(0 + 0 + 0 + 0 - 0 - 0)E[x] + s_1^2 E[v_1^2] + 0 + s_2^2 E[v_2^2] \\
&= s_1^2 E[v_1^2] + s_2^2 E[v_2^2] \\
&= s_1^2 \sigma_1^2 + s_2^2 \sigma_2^2 \\
&= s_1^2 \sigma_1^2 + (1 - s_1)^2 \sigma_2^2
\end{aligned}$$

## Mean Square Error

$$E[e^2] = s_1^2\sigma_1^2 + (1-s_1)^2\sigma_2^2$$

## Minimize the mean square error

$$E[e^2] = s_1^2\sigma_1^2 + (1-s_1)^2\sigma_2^2$$

$$\begin{aligned}\frac{\mathrm{dE[e^2]}}{\mathrm{ds_1}} &= 2s_1\sigma_1^2 - 2(1-s_1)\sigma_2^2 \\ &= 2s_1\sigma_1^2 + 2s_1\sigma_2^2 - 2\sigma_2^2 \\ &= 2s_1(\sigma_1^2+\sigma_2^2) - 2\sigma_2^2 = 0\end{aligned}$$

## Finding $S_1$

$$2s_1(\sigma_1^2+\sigma_2^2) - 2\sigma_2^2 = 0$$

$$2s_1(\sigma_1^2+\sigma_2^2) = 2\sigma_2^2$$

- Therefore

$$s_1 = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$$

## Finding $S_2$

$$\begin{aligned}s_2 &= 1 - s_1 \\ &= 1 - \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2} \\ &= \frac{\sigma_1^2+\sigma_2^2-\sigma_2^2}{\sigma_1^2+\sigma_2^2} \\ &= \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\end{aligned}$$

## Finally we get what we wanted

$$\begin{aligned}\hat{x} &= s_1 z_1 + s_2 z_2 \\ &= \left(\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right) z_1 + \left(\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right) z_2\end{aligned}$$

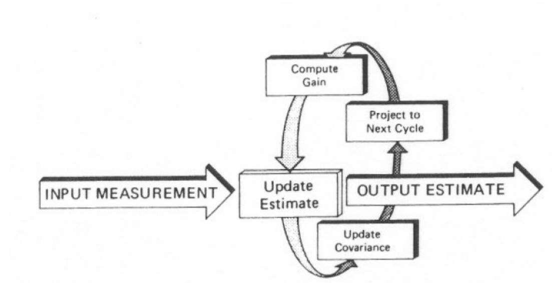
## Finding the new variance

$$\begin{aligned}\sigma^2 &= s_1^2\sigma_1^2 + s_2^1\sigma_2^2 \\ &= \left(\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)^2\sigma_1^2 + \left(\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2\sigma_2^2 \\ &= \frac{\sigma_2^4\sigma_1^2+\sigma_1^4\sigma_2^2}{\left(\sigma_1^2+\sigma_2^2\right)^2} \\ &= \frac{\sigma_1^2\sigma_2^2\left(\sigma_1^2+\sigma_2^2\right)}{\left(\sigma_1^2+\sigma_2^2\right)^2} \\ &= \frac{\sigma_1^2\sigma_2^2}{\left(\sigma_1^2+\sigma_2^2\right)} \\ &= \frac{1}{\left(\frac{\sigma_1^2+\sigma_2^2}{\sigma_2^2\sigma_1^2}\right)} \\ &= \frac{1}{\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}}\end{aligned}$$

## Formula for the new variance

$$\frac{1}{\sigma^2} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}$$

## Kalman Filter Diagram

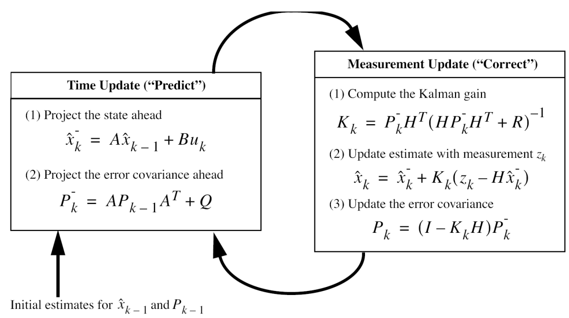



[Brown and Hwang (1992)]

## The process to be estimated

$$x_k = Ax_{k-1} + Bu_k + w_{k-1}$$

$$z_k = Hx_k + v_k$$

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