EE523: Random Processes for Communication and Signal Processing

Homework #5

1. A die is rolled repeatedly. Which of the following are Markov Chains. Find their transition matrix.
   a) The largest number $X_n$ shown up to the $n^{th}$ roll.
   b) The number $N_n$ of sixes in $n$ rolls.
   c) At time $r$, the time $C_r$ since the most recent six.
   d) At time $r$, the time $B_r$ until the next six.

2. Let $\{X_n, n \geq 1\}$ be i.i.d. integer-valued random variables. Let $S_n = \sum_{r=1}^{n} X_r$, with $S_0 = 0, Y_n = X_n + X_{n-1}$ with $X_0 = 0$ and $Z_n = \sum_{r=0}^{n} S_r$. Which of the following are Markov chains (a) $S_n$, (b) $Y_n$, (c) $Z_n$, and (d) the sequence of pairs $(S_n, Z_n)$.

3. Let $X$ be a Markov chain with a state $s$ that is absorbing, i.e. $p_{ss}(1) = 1$. All other states communicate with $s$ i.e. $i \rightarrow s$ for all states $i \in S$. Show that all states in $S$ except $s$ are transient.

4. Classify the states of the following Markov chains with $S = \{1, 2, 3, 4\}$ and transition matrices
   a) \[
   \begin{pmatrix}
   1/3 & 2/3 & 0 & 0 \\
   1/3 & 0 & 1/3 & 1/3 \\
   0 & 0 & 0 & 1 \\
   
   \end{pmatrix}
   \]
   b) \[
   \begin{pmatrix}
   0 & 1/2 & 1/2 & 0 \\
   1/3 & 0 & 0 & 2/3 \\
   1 & 0 & 0 & 0 \\
   0 & 0 & 1 & 0 \\
   \end{pmatrix}
   \]
   In case (a) calculate $f_{34}(n)$ and deduce that the probability of ultimate absorption in state 4, starting from 3 equals $\frac{2}{3}$.

5. Let $\{X_n, n \geq 0\}$ be a Markov chain with $X_0 = i$. Let $N$ be the total number of visits made by the chain to $j$. Show that
   \[
   P(N = n) = \begin{cases} 
   1 - f_{ij} & \text{if } n = 0 \\
   f_{ij}(f_{jj})^{n-1}(1 - f_{jj}) & \text{if } n \geq 1.
   \end{cases}
   \]
   and deduce that $P(N = \infty) = 1$ if and only if $f_{ij} = f_{jj} = 1$.

6. A particle performs a random walk on the vertices of a cube. At each step it remains where it is with probability $1/4$, or moves to one of the neighboring vertices with probability $1/4$. Let $v$ and $w$ be two diametrically opposite vertices. If the walk starts at $v$, find
   a) the mean number of steps until its first return to $v$.
   b) the mean number of steps until its first return to $w$. 


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