1. Show that if \( \text{var}(X) = 0 \) then \( X \) is almost surely constant. i.e. there exists a constant \( a \), such that \( P(X = a) = 1 \).

2. If \( X \) takes non-negative integer values, show that

\[
E(X) = \sum_{n=0}^{\infty} P(X > n)
\]

3. There are certain cases when our intuition about expected value may not be accurate.

(a) Give an example of a discrete random variable \( X \) whose expectation \( E(X) = \sum_x x P(X = x) \) is infinite. Note that you cannot assign a finite probability to \( X = \infty \) i.e. \( X \) is such that \( P(X < \infty) = 1 \).

(b) Give an example of a discrete random variable \( X \) such that \( x_{ML} = \arg \max_x P(X = x) \) is different from \( E(X) \).

4. Let \( \mathbf{X} = (X_1, X_2, \ldots, X_n)^T \) be a column vector of random variables. The covariance matrix of \( \mathbf{X} \) is given by

\[
\text{cov}(\mathbf{X}) = E\{(\mathbf{X} - E\mathbf{X})(\mathbf{X} - E\mathbf{X})^T\}
\]
i.e. it is a square matrix of size \( n \times n \). Suppose that the determinant of \( \text{cov}(\mathbf{X}) \) equals 0. Show that there exists \( a_i, i = 1, \ldots, n \) and \( b \) such that

\[
P\left(\sum_{i=1}^{n} a_i X_i = b\right) = 1
\]

*Hint: If \( \text{det}(A) = 0 \) then there exists a vector \( u \) such that \( u^T Au = 0 \).*

5. Suppose you buy a package every day. Suppose that there are \( c \) different types of objects and each package contains one of those objects. A package is equally likely to contain any of the \( c \) objects. Find the expected number of days that elapse before you have a full set of objects.

6. Let \( X \) and \( Y \) be random variables with mean 0 and variance 1 and \( E(XY) = \rho \). Show that

\[
E(\max(X^2, Y^2)) \leq 1 + \sqrt{1 - \rho^2}
\]

7. Show the following

(a) \( E(aY + bZ|X) = aE(Y|X) + bE(Z|X) \)

(b) If \( X \) and \( Y \) are independent \( E(Y|X) = E(Y) \)

(c) \( E(E(Y|X,Z)|X) = E(Y|X) = E(E(Y|X)|X,Z) \)
8. The conditional variance of $Y$ given $X$, $\text{var}(Y|X)$ is defined by

$$\text{var}(Y|X = x) = \mathbb{E}((Y - \mathbb{E}(Y|X = x))^2 | X = x)$$

Note that the conditional variance is a random variable. Show that

$$\text{var}(Y) = \mathbb{E}(\text{var}(Y|X)) + \text{var}(\mathbb{E}(Y|X))$$