KF-CS Theory

Animesh Biswas, Kevin Palmowski

Last compiled June 11, 2014

1 Model

At each time t, we have $y_t = Ax_t + w_t$.

Time indices are discrete. Make the distinction between sampling times (used) and continuous time (not used).

2 Algorithm – KF-CS with LS

3 Proofs

Lemma 1. *Assume that $\{x_t\}$ follow the signal model above, $y_t = Ax_t + w_t$, $\{t_0, t_0+1, t_0+2, \ldots\}$ is a discrete set of sampling times, only additions to true support ($N_t \subseteq N_{t+1}$ for all t), etc.*

Further assume that

i) The true solution is exactly recovered at the initial time t_0: $\hat{x}_{t_0} = x_{t_0}$, so $\hat{N}_{t_0} = N_{t_0} = N_0$; ***Can we relax this to just the true support is recovered?***

ii) The maximum support size S_{max} satisfies $S_{max} \le S_{**} = \max\{s : \delta_{2s}(A) < \sqrt{2}-1\}$;

iii) The observation noise w_t is bounded in magnitude: $\|w_t\| < \xi$ for all t and some $\xi > 0$;

iv) The addition threshold α_t satisfies $\alpha_t = C_1\xi$ for each sampling time t, where $C_1 = C_1(|N_t|, \xi)$ ***(verify)*** *is defined* ***below OR in Candes****; and*

v) The addition delay d satisfies $d > \tau_{det}$, where

$$\tau_{det} = \tau_{det}(\alpha_t, \varepsilon) = \left\lceil \left(\frac{2\alpha_t}{\sigma_{sys}\mathbf{\Phi}^{-1}\left(\frac{(1-\varepsilon)^{1/S_a}}{2}\right)} \right)^2 \right\rceil . \leftarrow \textit{\textbf{superscript looks bad}}$$

Here, $\mathbf{\Phi}^{-1}(x)$ is the inverse of the standard Guassian CDF, $\mathbf{\Phi}(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dz$.

Then

1) $\|x_t - \hat{x}_{t,CSres}\|_2 \le \alpha_t$ for all sampling times t;

2) $\hat{N}_t \subseteq N_t$ for all sampling times t; and

3) $\textbf{\textit{Pr}}(E_j \mid F_j) \ge 1 - \varepsilon$, where $E_j = \{\hat{N}_t = N_t \text{ for all } t \in [t_j + \tau_{det} : t_{j+1} - 1]\}$ and $F_j = \{\hat{N}_{t_j-1} = N_{t_j-1}\}$.

Proof. **Need to find some way to get Candes Thm 1.3 in here and make the connection that $\hat{x}_{t,\mathbf{CSres}}$ in our notation is $x^\star$ in his**

To prove claims 1 and 2, we proceed by induction on the value of t.

Consider the base case, where $t = t_0$. Claim 1 follows from Theorem 1.3 in [1] and assumptions (ii), (iii), and (iv) **(Not immediate – need to connect to Candes as above)**, and assumption (i) trivially proves claim 2.

Suppose now that claims 1 and 2 are true for some time $(t-1)$. We show that the claims are true at time t.

First, we verify claim 1 at time t. Let

$$\begin{aligned} \beta_t &= x_t - \hat{x}_{t,\text{init}} \\ \hat{\beta}_t &= \arg\min_{\beta} \|\beta\|_1 \text{ subject to } \|y_t - A\beta\|_2 < \xi \\ \hat{x}_{t,\text{CSres}} &= \hat{x}_{t,\text{init}} + \hat{\beta}_t, \end{aligned}$$

where $\hat{x}_{t,\text{init}}$ is defined in the algorithm and $\text{supp}(\hat{x}_{t,\text{init}}) = \hat{N}_{t-1}$.

By the induction hypothesis, $\hat{N}_{t-1} \subseteq N_{t-1}$, and by our model assumptions we have $N_{t-1} \subseteq N_t$. Therefore, $\text{supp}(\beta_t) \subseteq N_t \cup N_{t-1} = N_t$, so $|\,\text{supp}(\beta_t)| \leq |N_t| \leq S_{\max}$. With this, we can apply Theorem 1.3 in [1] to see that $\|\beta_t - \hat{\beta}_t\|_2 \leq \alpha_t$ **(AGAIN, need to make this connection)**. By the definitions of β_t and $\hat{x}_{t,\text{CSres}}$, we see that $\|\beta_t - \hat{\beta}_t\|_2 = \|x_t - \hat{x}_{t,\text{CSres}}\|_2$, so claim 1 follows.

Next, we verify claim 2 at time t. Suppose that $(x_t)_i = 0$ for some index i, so that $i \notin \text{supp}(x_t) = N_t$. Since $N_{t-1} \subseteq N_t$, we must also have $i \notin N_{t-1}$; by the induction hypothesis, this implies that $i \notin \hat{N}_{t-1}$.

Applying the result of claim 1,

$$|(\hat{x}_{t,\text{CSres}})_i|^2 = |(x_t - \hat{x}_{t,\text{CSres}})_i|^2 \leq \|x_t - \hat{x}_{t,\text{CSres}}\|_2^2 \leq \alpha_t^2,$$

so $|(\hat{x}_{t,\text{CSres}})_i| \leq \alpha_t$. Referring to the algorithm, $\hat{N}_t = \hat{N}_{t-1} \cup \{j : |(\hat{x}_{t,\text{CSres}})_j| > \alpha_t\}$. Since $i \notin \hat{N}_{t-1}$ and $|(\hat{x}_{t,\text{CSres}})_i| \leq \alpha_t$, it follows that $i \notin \hat{N}_t$. Thus if $i \notin N_t$, then $i \notin \hat{N}_t$; equivalently, if $i \in \hat{N}_t$, then $i \in N_t$. Therefore, $\hat{N}_t \subseteq N_t$, which proves claim 2 and completes our induction proof.

Now, we prove claim 3. **FINISH THIS**

□

References

[1] Emmanuel J. Candès, *The restricted isometry property and its implications for compressed sensing*, Comptes Rendus Mathematique, Volume 346, Issues 9–10, May 2008, Pages 589-592, ISSN 1631-073X, http://dx.doi.org/10.1016/j.crma.2008.03.014.