

KFCS Theory

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1 Model

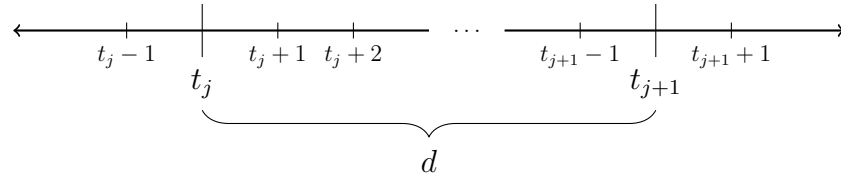
At each time $t \geq t_0$, we have

$$\begin{aligned} y_t &= Ax_t + w_t \\ x_{t+1} &= x_t + \nu_{t+1} \end{aligned} \quad \text{Can we use } \nu_t?$$

Here, $\mathbb{E}[w_t] = \mathbf{0}$, $\text{cov}(w_t) = \mathbb{E}[w_t w_t'] = R = \sigma_{\text{obs}}^2 I_{n \times n}$, iid and independent of x_t ; $x_{t_0} = x_0 \sim \mathcal{N}(\mathbf{0}, \sigma_{\text{sys},0}^2 I_{N_0})$; and $\nu_t \sim \mathcal{N}(\mathbf{0}, \sigma_{\text{sys}}^2 I_{N_t})$ iid.

$y_t, w_t \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times m}$, $x_t, \nu_t \in \mathbb{R}^m$.

Time indices are discrete. Make the distinction between sampling times (used) and continuous time (not used).



At the addition times $t_j = t_0 + jd$ for some t_0 , the support of x_t changes: $N_t = N_{t_j}$ for all $t \in [t_j : t_{j+1} - 1]$, and $N_{t_j} \subset N_{t_{j+1}}$.

2 Algorithm – KFCS with LS

This algorithm applies to the case where there are no support deletions.

Issues:

P_{t_0-1} and Q_t – is this an identity of size $|N_{hat}|$ or is it a full-blown identity with nonzeros on diagonal entries corresponding to N_{hat}

Is this algorithm transcribed correctly? There are 3 versions of it that I have (NV original, AB+NV typed draft, and AB handwritten) and all 3 are different.

Look for places to simplify – this is long and contains repeat steps, which is non-ideal

Input: $\sigma_{\text{sys}}, \sigma_{\text{obs}}, \sigma_{\text{sys},0}, A, \{t_j\}, \{N_t\}, \{y_t\}$

$$\hat{x}_{t_0, \text{init}} = \arg \min_x \|x\|_1 \text{ subject to } \|y_{t_0} - Ax\|_2 < \xi$$

$$\hat{N}_{t_0} = \{j : |(\hat{x}_{t_0, \text{init}})_j| > \alpha\}$$

$$P_{t_0-1} = \sigma_{\text{sys},0}^2 I_{\hat{N}_{t_0}}$$

$$Q_{t_0} = 0$$

$$\hat{x}_{t_0-1} = \mathbf{0}$$

$$P_{t_0|t_0-1} = P_{t_0-1} + Q_{t_0}$$

$$K_{t_0} = P_{t_0|t_0-1} A' (AP_{t_0|t_0-1} A' + \sigma_{\text{obs}}^2 I)^{-1}$$

$$J_{t_0} = I - K_{t_0} A$$

$$P_{t_0} = J_{t_0} P_{t_0|t_0-1}$$

$$\hat{x}_{t_0} = J_{t_0} \hat{x}_{t_0-1} + K_{t_0} y_{t_0}$$

for $t > t_0$ **do**

$$Q_t = \sigma_{\text{sys}}^2 I_{\hat{N}_{t-1}}$$

$$P_{t|t-1} = P_{t-1} + Q_t$$

$$K_t = P_{t|t-1} A' (AP_{t|t-1} A' + \sigma_{\text{obs}}^2 I)^{-1}$$

$$J_t = I - K_t A$$

$$P_t = J_t P_{t|t-1}$$

$$\hat{x}_{t, \text{init}} = J_t \hat{x}_{t-1} + K_t y_t$$

$$y_{t, \text{res}} = y_t - A \hat{x}_{t, \text{init}}$$

$$\hat{\beta}_t = \arg \min_{\beta} \|\beta\|_1 \text{ subject to } \|y_{t, \text{res}} - A\beta\|_2 < \xi$$

$$\hat{x}_{t, \text{CSres}} = \hat{x}_{t, \text{init}} + \hat{\beta}_t$$

$$\Delta_A = \{j : |(\hat{x}_{t, \text{CSres}})_j| > \alpha\}$$

$$\hat{N}_t = \hat{N}_{t-1} \cup \Delta_A$$

if $\Delta_A = \emptyset$ **then**

$$| \hat{x}_t = \hat{x}_{t, \text{init}}$$

else

$$| \hat{x}_t = \mathbf{0}$$

$$| (\hat{x}_t)_{\hat{N}_t} = (A_{[1:n], \hat{N}_t})^\dagger y_t$$

$$| P_t = 0_{m \times m}$$

$$| (P_t)_{\hat{N}_t, \hat{N}_t} = \left[(A_{[1:n], \hat{N}_t})' (A_{[1:n], \hat{N}_t}) \right]^{-1} \sigma_{\text{obs}}^2 I_{|\hat{N}_t|}$$

end

end

3 Algorithm – Genie-Aided Kalman Filtering (GAKF)

This algorithm applies to the case where there are no support deletions.

Issues:

Check blue piece below – do we want all-ones, identity of size $|\Delta A|$, or identity restricted to ΔA and zero else?

Input: $\sigma_{\text{sys}}, \sigma_{\text{obs}}, \sigma_{\text{sys},0}, A, \{t_j\}, \{N_t\}, \{y_t\}$

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for  $t \geq t_0$  do
  if  $t = t_0$  then
     $T = N_0$ 
     $\tilde{P}_{t-1} = \sigma_{\text{sys},0}^2 I_T$ 
     $\tilde{x}_{t-1} = \mathbf{0}$ 
     $\tilde{Q}_t = 0$ 
  else
     $T = N_{t-1}$ 
     $\tilde{Q}_t = \sigma_{\text{sys}}^2 I_T$ 
    if  $t = t_j$  for some  $j > 0$  then
       $\Delta_A = N_t \setminus N_{t-1}$ 
       $\left( \tilde{P}_{t-1} \right)_{\Delta_A, \Delta_A} = \sigma_{\text{sys}}^2 I_{|\Delta_A|}$ 
    end
  end
   $\tilde{P}_{t|t-1} = \tilde{P}_{t-1} + \tilde{Q}_t$ 
   $\tilde{K}_t = \tilde{P}_{t|t-1} A' \left( A \tilde{P}_{t|t-1} A' + \sigma_{\text{obs}}^2 I \right)^{-1}$ 
   $\tilde{J}_t = I - \tilde{K}_t A$ 
   $\tilde{P}_t = \tilde{J}_t \tilde{P}_{t|t-1}$ 
   $\tilde{x}_t = \tilde{J}_t \tilde{x}_{t-1} + \tilde{K}_t y_t$ 
end

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4 Candes RIP – C_1 Computation for α

We need to add this as a theorem or something – cite [1] Thm 1.3 and explicitly give the value of C_1 and the commentary below.

THEOREM / RESULT: [1], Theorem 1.3

Suppose $y = Ax + \eta$, $|\text{supp}(x)| = s$, $\delta_{2s} = \delta_{2s}(A) < \sqrt{2} - 1$, and $\|\eta\|_2 \leq \xi$. Then

$$\hat{x} = \arg \min_z \|z\|_1 \text{ subject to } \|y - Az\|_2 \leq \xi$$

satisfies

$$\|x - \hat{x}\|_2 \leq C_1(s)\xi,$$

where

$$C_1(s) = \frac{4\sqrt{1 + \delta_{2s}}}{1 - (1 + \sqrt{2})\delta_{2s}}.$$

Claim / Note: It can be shown that C_1 is an increasing function of δ_{2s} , and δ_{2s} is an increasing function of s , so C_1 is an increasing function of s .

For any support size S in this paper, we will have $S \leq S_{\max}$ and thus $C_1(S) \leq C_1(S_{\max})$.

5 Linear Systems Theory

5.1 Definitions

We present some basic definitions from linear systems theory. These can be found in [?], Appendix C. **matrix sizes / rank = n, etc.**

A matrix F is **stable** if $\rho(F) < 1$.

The pair $\{F, G\}$ is **controllable** if the matrix $[G, FG, \dots, F^{n-1}G]$ is full rank n . An equivalent characterization of observability is that $\text{rank}([\lambda I - F, G]) = n$ for all eigenvalues λ of F .

The pair $\{F, G\}$ is **unit-circle controllable** if $\text{rank}([\lambda I - F, G]) = n$ for all eigenvalues λ of F with $|\lambda| = 1$.

The pair $\{F, G\}$ is **stabilizable** if $\text{rank}([\lambda I - F, G]) = n$ for all eigenvalues λ of F with $|\lambda| \geq 1$.

The pair $\{F, H\}$ is **detectable** if and only if $\{F', H'\}$ is stabilizable.

Consider the case where $F = I$. Then $\lambda = 1$ is the only eigenvalue of $F = F'$ and the matrix $[\lambda I - F, G] = [0, G]$ has rank n if and only if G has rank n . Therefore, if G is full rank, then $\{I, G\}$ is controllable, unit-circle controllable, and stabilizable. Additionally, since $\text{rank}(H) = \text{rank}(H')$, we can use the same argument to conclude that $\{I, H\}$ is detectable if H is full rank.

5.2 Theoretical Results

Here we present two important theoretical results from linear systems theory. **Specialize these results for our case where $S = 0$, etc.?**

The general form of a **discrete-time algebraic Riccati equation (DARE)** is

$$P = FPF' + GQG' - (FPH' + GS)(R + HPH')^{-1}(FPH' + GS)'. \quad (1)$$

Define $F^s = F - GSR^{-1}H$ and $Q^s = Q - SR^{-1}S'$.

[3], **Theorem E.5.1:** Consider the discrete-time algebraic Riccati equation (1). The following two statements are equivalent: (i) $\{F, H\}$ is detectable and $\{F^s, GQ^{s/2}\}$ is controllable on the unit circle. (ii) The DARE has a stabilizing solution P , i.e., one for which the matrix $F - K_p H$ is stable, where $K_p = (FPH' + GS)(R + HPH')^{-1}$. Moreover, any such stabilizing solution is unique and positive semi-definite.

[2], **Lemma 8.7.3:** Consider the Riccati recursion with positive semi-definite initial condition

$$P_{i+1} = FP_iF' + GQG' - K_{p,i}R_{e,i}K'_{p,i}, \quad P_0 \succeq 0.$$

If $Q \succ 0$, $R \succ 0$, $\{F, H\}$ is detectable and $\{F^s, GQ^s\}$ is stabilizable then P_i converges to the unique positive semi-definite matrix, P , that satisfies the discrete-time algebraic Riccati equation (1).

ISSUE: Hassibi thesis requires $\{F^s, GQ^s\}$ stabilizable, not $GQ^{s/2}$. Inconsistent with KSH and NV draft.

6 Kailath, Sayed, Hassibi – Linear Estimation

TEMPORARY SECTION

6.1 Appendix C, Section 3

Definitions and concepts from system theory.

$$x_{i+1} = F_i x_i + G_i u_i$$

$$y_i = H_i x_i + K_i u_i$$

Shorthand (F_i, G_i, H_i, K_i) . **PROBLEM:** y is tied also to u here instead of some other variable...

Stable: F is stable if $|\lambda_i| < 1$ for all $\lambda_i \in \sigma_p(F)$, equivalently, $\rho(F) < 1$.

Controllable: $\{F, G\}$ is controllable if and only if the controllability matrix $\mathcal{C} = [G, FG, F^2G, \dots, F^{n-1}G]$ is full rank n .

Unit-circle Controllable: $\{F, G\}$ is unit-circle controllable if $\text{rank}(\lambda I - FG) = n$ at all unit-circle eigenvalues λ of F .

P. 502 – unit-circle controllable \equiv there exists K with $F - GK$ has no unit-circle eigenvalues. Is this even true? Nontrivial if so.

Stabilizable: $\{F, G\}$ is stabilizable if and only if $\text{rank}(\lambda I - FG) = n$ for all $\lambda \in \sigma_p(F)$ with $|\lambda| \geq 1$.

Observable: $\{F, H\}$ is observable if and only if $\{F^*, H^*\}$ is controllable, i.e. $\mathcal{C} = [H^*, F^*H^*, (F^*)^2H^*, \dots, (F^*)^{n-1}H^*]$ is full rank n .

Detectable: $\{F, H\}$ is detectable if and only if $\{F^*, H^*\}$ is stabilizable, i.e. $\text{rank}(\lambda I - F^*H^*) = n$ for all $\lambda \in \sigma_p(F)$ with $|\lambda| \geq 1$.

6.2 Chapter 8, section 3

Solutions of a DARE

Section 8.1 – Time-Invariant State-Space Model (p.266)

$$x_{i+1} = Fx_i + Gu_i$$

$$y_i = Hx_i + v_i$$

$$Q = \text{cov}(u), \quad R = \text{cov}(v), \quad S = \text{cov}(u, v)$$

Discrete Algebraic Riccati Equation (DARE):

$$P = FPF^* + GQG^* - K_p R_e K_p^*$$

$$R_e = R + HPH^*$$

$$K_p = (FPH^* + GS)R_e^{-1}$$

Define $F^s = F - GSR^{-1}H$ and $Q^s = Q - SR^{-1}S^*$.

Theorem 8.3.1: Assume that F is stable (otherwise, $\{F, H\}$ is detectable), $\{F^s, GQ^{s/2}\}$ controllable on the unit circle,

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \succeq 0, \quad R \succ 0.$$

Under these conditions, the DARE has a unique solution P such that $F - K_p H$ is stable. Moreover, this so-called stabilizing solution is positive semi-definite and results in a positive definite $R_e = R + HPH^*$.

6.3 Chapter 14

Lemma 14.2.1: Consider the zero-initial-condition Riccati recursion and assume $\{F, H\}$ detectable and $\{F, GQ^{1/2}\}$ unit-circle controllable so that the unique stabilizing solution P exists. Then P_i^0 converges to P if and only if $\{F, GQ^{1/2}\}$ is stabilizable.

I don't think we're in the zero-initial-condition case; rather, $P^0 = \sigma I$.

Exercise 14.4? p.546

6.4 Appendix E

(Confirm the statement – p.783)

Theorem E.5.1: Consider the DARE (above). The following are equivalent: (i) $\{F, H\}$ is detectable and $\{F^s, GQ^{s/2}\}$ is controllable on the unit circle; (ii) the DARE has a stabilizing solution P , i.e., one for which the matrix $F - K_p H$ is stable. Moreover, any such stabilizing solution is unique and positive semi-definite.

7 Callier / Desoer – Linear System Theory

TEMPORARY SECTION

Model – p. 57 – discrete-time system representation

$$x_{k+1} = A_k x_k + B_k u_k$$

$$y_k = C_k x_k + D_k u_k$$

Problem: we have w_k instead of $D_k u_k$. We can always find an appropriate D_k unless $u_k = 0$, which happens with probability 0, presumably, so it should be okay?

Time-invariant: (A, B, C, D) constant in time.

(Thms: pp. 293–294)

Theorem 8.d.62.ii Consider a discrete-time time-invariant system representation $R_d = [A, B, C, D]$. If A and B are real, there exists $F \in \mathbb{R}^{n_i \times n}$ such that

$$\sigma_p(A + BF) \subset \mathcal{D}(0, 1) = \{z \in \mathbb{C} : |z| < 1\}$$

if and only if the pair (A, B) is stabilizable.

Theorem 8.d.65.ii Consider a discrete-time time-invariant system representation $R_d = [A, B, C, D]$. If C and A are real, there exists $L \in \mathbb{R}^{n \times n_o}$ such that

$$\sigma_p(A + LC) \subset \mathcal{D}(0, 1) = \{z \in \mathbb{C} : |z| < 1\}$$

if and only if the pair (C, A) is detectable.

Problem: These theorems are for a time-invariant system. We have time-invariance on everything except D_k due to our w_k issue. This may be a non-issue, but I have no idea.

Translate to (F, G, H, K) notation:

Detectable: $\{F, H\}$ is detectable if and only if $\rho(F + LH) < 1$ for some L .

Stabilizable: $\{F, G\}$ is stabilizable if and only if $\rho(F + GL) < 1$ for some L .

Translate to our problem, $F = I$, $H = A$, $G = I$, $Q = \sigma_{\text{sys}}^2 I$:

Detectable: $\{I, A\}$ is detectable if and only if $\rho(I + LA) < 1$ for some L .

Stabilizable: $\{I, \sigma_{\text{sys}} I\}$ is stabilizable if and only if $\rho(I + \sigma_{\text{sys}} L) < 1$ for some L .

8 Hassibi – PhD Thesis

TEMPORARY SECTION

Results on Riccati equations.

Section 7.3, footnote 1: Def: $\{F, H\}$ is detectable if there exists a matrix K such that $F - KH$ is stable.

Linking this with the definition of detectable from KSHLinear:

$$\exists K \text{ with } F - KH \text{ stable} \equiv \text{detectable} \equiv \{F^*, H^*\} \text{ stabilizable}$$

Lemma 8.7.3 Consider the Riccati recursion with positive semi-definite initial condition

$$P_{i+1} = FP_iF^* + GQG^* - K_{p,i}R_{e,i}K_{p,i}^*, \quad P_0 \succeq 0.$$

If $Q \succ 0$, $R \succ 0$, $\{F, H\}$ is detectable and $\{F - GSR^{-1}H, GQ - GSR^{-1}S^*\}$ is stabilizable then P_i converges to the unique positive semi-definite matrix, P , that satisfies the discrete-time algebraic Riccati equation

$$P = FPF^* + GQG^* - (FPH^* + GS)(R + HPH^*)^{-1}(FPH^* + GS)^*.$$

Theorem 8.7.1 Consider the Riccati recursion

$$P_{i+1} = FP_iF^* + GQG^* - K_{p,i}R_{e,i}K_{p,i}^*, \quad P_0$$

where $R \succ 0$, $Q - SR^{-1}S^* \succ 0$, $\{F, H\}$ is detectable and $\{F - GSR^{-1}H, GQ - GSR^{-1}S^*\}$ is stabilizable. Suppose, moreover, that the initial condition P_0 is such that

$$I + (P^a)^{*/2}P_0(P^a)^{1/2} \succ 0,$$

where P^a is the unique positive semi-definite solution to the dual Riccati recursion. Then P_i converges to the unique positive semi-definite matrix, P , that satisfies the discrete-time algebraic Riccati equation

$$P = FPF^* + GQG^* - (FPH^* + GS)(R + HPH^*)^{-1}(FPH^* + GS)^*.$$

9 Proofs

Lemma 1. Assume that $\{x_t\}$ and $\{y_t\}$ follow the signal model above, $\{t_0, t_0 + 1, t_0 + 2, \dots\}$ is a discrete set of sampling times, only additions to true support ($N_t \subseteq N_{t+1}$ for all t), etc.

Further assume that

- i) The true solution is exactly recovered at the initial time t_0 : $\hat{x}_{t_0} = x_{t_0}$, so $\hat{N}_{t_0} = N_{t_0} = N_0$; *Can we relax this to just the true support is recovered?*
- ii) The maximum support size S_{max} satisfies $S_{max} \leq S_{**} = \max\{s : \delta_{2s}(A) < \sqrt{2} - 1\}$;
- iii) The observation noise w_t is bounded in magnitude: $\|w_t\| < \xi$ for all t and some $\xi > 0$;
- iv) The addition thresholds α_t satisfy $\alpha_t = \alpha = C\xi$ for all t , where

$$C = C(S_{max}) = \frac{4\sqrt{1 + \delta_{2S_{max}}}}{1 - (1 + \sqrt{2})\delta_{2S_{max}}}$$

with $\delta_{2S_{max}} = \delta_{2S_{max}}(A)$; and

- v) The addition delay d satisfies $d > \tau_{det}$, where the detection delay τ_{det} is defined by

$$\tau_{det} = \tau_{det}(\alpha, \varepsilon) = \left\lceil \left(\frac{2\alpha}{\sigma_{sys} \mathcal{Q}^{-1}\left(\frac{(1-\varepsilon)^{1/S_{add}}}{2}\right)} \right)^2 \right\rceil.$$

Here, $\mathcal{Q}^{-1}(x)$ is the inverse of the Gaussian \mathcal{Q} -function, $\mathcal{Q}(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$.

Then

- 1) $\|x_t - \hat{x}_{t, CSres}\|_2 \leq \alpha$ for all sampling times t ;
- 2) there are no false support additions: $\hat{N}_t \subseteq N_t$ for all sampling times t ; and
- 3) $\Pr(\mathbf{E}_j | \mathbf{F}_j) \geq 1 - \varepsilon$, where $\mathbf{E}_j = \{\hat{N}_t = N_t \text{ for all } t \in [t_j + \tau_{det} : t_{j+1} - 1]\}$ and $\mathbf{F}_j = \{\hat{N}_{t_j-1} = N_{t_j-1}\}$.

We may want to split claim 3 into its own piece because its proof relies on the other 2 parts, which are proved separately with induction.

Proof. Need to find some way to get Candes Thm 1.3 in here and make the connection that $\hat{x}_{t,\text{CSres}}$ in our notation is x^* in his. Also need to point out that the way we chose α , we have any $C_1\xi \leq C_1(S_{\max})\xi = \alpha$.

To prove claims 1 and 2, we proceed by induction on the value of t .

Consider the base case, where $t = t_0$. Claim 1 follows from Theorem 1.3 in [1] and assumptions (ii), (iii), and (iv) (**Not immediate – need to connect to Candes as above**), and assumption (i) trivially proves claim 2.

Suppose now that claims 1 and 2 are both true for some time $(t - 1)$. We show that the claims are true at time t .

First, we verify claim 1 at time t . Let **TYP0 HERE?** $y_t \rightarrow y_{t,\text{res}}$? **Can probably remove after getting algorithm typed up.**

$$\begin{aligned}\beta_t &= x_t - \hat{x}_{t,\text{init}} \\ \hat{\beta}_t &= \arg \min_{\beta} \|\beta\|_1 \text{ subject to } \|y_{t,\text{res}} - A\beta\|_2 < \xi \\ \hat{x}_{t,\text{CSres}} &= \hat{x}_{t,\text{init}} + \hat{\beta}_t,\end{aligned}$$

where $\hat{x}_{t,\text{init}}$ and $y_{t,\text{res}}$ are defined in the KFCS with LS algorithm and $\hat{x}_{t,\text{init}}$ satisfies $\text{supp}(\hat{x}_{t,\text{init}}) = \hat{N}_{t-1}$.

By the induction hypothesis, $\hat{N}_{t-1} \subseteq N_{t-1}$, and by our model assumptions we have $N_{t-1} \subseteq N_t$. Therefore, $\text{supp}(\beta_t) \subseteq N_t \cup N_{t-1} = N_t$, so $|\text{supp}(\beta_t)| \leq |N_t| \leq S_{\max}$. With this, we can apply Theorem 1.3 in [1] to see that $\|\beta_t - \hat{\beta}_t\|_2 \leq \alpha$ (**AGAIN, need to make this connection**). By the definitions of β_t and $\hat{x}_{t,\text{CSres}}$, we see that $\|\beta_t - \hat{\beta}_t\|_2 = \|x_t - \hat{x}_{t,\text{CSres}}\|_2$, so claim 1 follows.

Next, we verify claim 2 at time t . Suppose that $(x_t)_i = 0$ for some index i , so that $i \notin \text{supp}(x_t) = N_t$. Since $N_{t-1} \subseteq N_t$, we must also have $i \notin N_{t-1}$; by the induction hypothesis, this implies that $i \notin \hat{N}_{t-1}$.

Applying the result of claim 1,

$$|(\hat{x}_{t,\text{CSres}})_i| = |(x_t - \hat{x}_{t,\text{CSres}})_i| \leq \|x_t - \hat{x}_{t,\text{CSres}}\|_2 \leq \alpha.$$

Referring to the algorithm, $\hat{N}_t = \hat{N}_{t-1} \cup \{j : |(\hat{x}_{t,\text{CSres}})_j| > \alpha\}$. Since $i \notin \hat{N}_{t-1}$ and $|(\hat{x}_{t,\text{CSres}})_i| \leq \alpha$, it follows that $i \notin \hat{N}_t$. Thus if $i \notin N_t$, then $i \notin \hat{N}_t$; equivalently, if $i \in \hat{N}_t$, then $i \in N_t$. Therefore, $\hat{N}_t \subseteq N_t$, which proves claim 2 and completes our induction proof.

Now, we prove claim 3. Let $\Delta_t = N_t \setminus \hat{N}_{t-1}$ denote the set of indices of the true support at time t which have not been detected before time t . Suppose that \mathbf{F}_j holds, that is, $\hat{N}_{t_j-1} = N_{t_j-1}$.

Since \mathbf{F}_j holds, $\Delta_t \subseteq \Delta_{\text{add}, t_j}$ for all $t \in [t_j : t_{j+1} - 1]$.

Let $i \in \Delta_t$ for some $t \in [t_j : t_{j+1} - 1]$ and suppose that $|(x_t)_i| > 2\alpha$. Applying the result from claim 1,

$$0 \leq |(x_t - \hat{x}_{t, \text{CSres}})_i| \leq \|(x_t - \hat{x}_{t, \text{CSres}})\|_2 \leq \alpha < 2\alpha < |(x_t)_i|,$$

so that

$$\begin{aligned} |(\hat{x}_{t, \text{CSres}})_i| &= |(x_t)_i - [(x_t)_i - (\hat{x}_{t, \text{CSres}})_i]| \\ &\geq ||(x_t)_i| - |(x_t - \hat{x}_{t, \text{CSres}})_i|| \\ &= |(x_t)_i| - |(x_t - \hat{x}_{t, \text{CSres}})_i| \\ &> 2\alpha - \alpha \\ &= \alpha. \end{aligned}$$

We see that if $|(x_t)_i| > 2\alpha$, then $|(\hat{x}_{t, \text{CSres}})_i| > \alpha$, so $i \in \hat{N}_t = \hat{N}_{t-1} \cup \{j : |(\hat{x}_{t, \text{CSres}})_j| > \alpha\}$.

If $|(x_t)_i| > 2\alpha$ for all $i \in \Delta_{\text{add}, t_j}$, then $\Delta_t \subseteq \Delta_{\text{add}, t_j} \subseteq \hat{N}_t$; in words, we will detect all “missing” indices at time t , so $\hat{N}_t = N_t$.

From the above discussion, we see that the event $\{|(x_t)_i| > 2\alpha \text{ for all } i \in \Delta_{\text{add}, t_j}\}$ is contained within the event $\{|(x_t)_i| > 2\alpha \text{ for all } i \in \Delta_t \mid \mathbf{F}_j\}$, which in turn is contained within the event $\{\hat{N}_t = N_t \mid \mathbf{F}_j\}$.

All of the above is still kind of weak in places. It all makes sense in words and is true, but the math / set theory is kind of wonky.

Our model asserts that the entries $(x_t)_i$ of x_t are independent and identically distributed $\mathcal{N}(0, (t - t_j)\sigma_{\text{sys}}^2)$ random variables. With this in mind, we see that

$$\begin{aligned} \Pr(\hat{N}_t = N_t \mid \mathbf{F}_j) &\geq \Pr(|(x_t)_i| > 2\alpha \text{ for all } i \in \Delta_t \mid \mathbf{F}_j) \\ &\geq \Pr(|(x_t)_i| > 2\alpha \text{ for all } i \in \Delta_{\text{add}, t_j}) \\ &= [\Pr(|(x_t)_1| > 2\alpha)]^{S_{\text{add}}} \\ &= \left[2\mathcal{Q}\left(\frac{2\alpha}{\sigma_{\text{sys}}\sqrt{t - t_j}}\right) \right]^{S_{\text{add}}}. \end{aligned}$$

We examine the particular case where $t = t_j + \tau_{\text{det}}$. In this case,

$$\begin{aligned} \mathbf{Pr} \left(\hat{N}_{t_j + \tau_{\text{det}}} = N_{t_j + \tau_{\text{det}}} \mid \mathbf{F}_{\mathbf{j}} \right) &\geq \left[2\mathcal{Q} \left(\frac{2\alpha}{\sigma_{\text{sys}} \sqrt{(t_j + \tau_{\text{det}}} - t_j)} \right) \right]^{S_{\text{add}}} \\ &= \left[2\mathcal{Q} \left(\frac{2\alpha}{\sigma_{\text{sys}} \sqrt{\tau_{\text{det}}}} \right) \right]^{S_{\text{add}}} \\ &\geq 1 - \varepsilon, \end{aligned}$$

where the final inequality is easily verified and follows from the ceiling in the definition of τ_{det} and the fact that \mathcal{Q} is a decreasing function.

If $\hat{N}_t = N_t$ for $t = t_j + \tau_{\text{det}}$, then the model assumptions of no support deletions and no support additions until time t_{j+1} , in addition to the result of claim 2, imply that $\hat{N}_t = N_t$ for all $t \in [t_j + \tau_{\text{det}} : t_{j+1} - 1]$, which is exactly the event $\mathbf{E}_{\mathbf{j}}$. Therefore, $\mathbf{Pr}(\mathbf{E}_{\mathbf{j}} \mid \mathbf{F}_{\mathbf{j}}) = \mathbf{Pr} \left(\hat{N}_{t_j + \tau_{\text{det}}} = N_{t_j + \tau_{\text{det}}} \mid \mathbf{F}_{\mathbf{j}} \right) \geq 1 - \varepsilon$, which completes the proof. \square

Lemma 2. Assume that $\{x_t\}$ and $\{y_t\}$ follow the signal model above, $\{t_0, t_0 + 1, t_0 + 2, \dots\}$ is a discrete set of sampling times, only additions to true support ($N_t \subseteq N_{t+1}$ for all t), etc.

$$\delta_{S_{\max}}(A) < 1, \alpha_{\text{del}} = 0.$$

Define the event $\mathbf{D} = \{\hat{N}_t = N_t = N_* \text{ for all } t \in [t_* : t_{**}]\}$, where N_* is some fixed index set.

At each time t , let $\hat{x}_t = \hat{x}_{t, \text{KFCS}}$ be the KFCS estimate of x_t and let $\tilde{x}_t = \hat{x}_{t, \text{GAKF}}$ be the GAKF estimate of x_t .

Then given any $\varepsilon > 0$ there exists some $t_{ms} \geq t_*$ such that for all $t \in [t_{ms} : t_{**}]$, we have $\mathbb{E}[\|\tilde{x}_t - \hat{x}_t\|_2^2 | \mathbf{D}] < \varepsilon$, i.e., \tilde{x}_t converges to \hat{x}_t in mean square.

Proof. Throughout, we assume that the event \mathbf{D} occurs and $t \in [t_* : t_{**}]$.

Where possible, we consider variables and parameters only along the support set N_* , but to simplify notation will omit the subscript N_* . Thus, $\nu_t = (\nu_t)_{N_*}$, $A = A_{[1:n], N_*}$, $Q = Q_{N_*, N_*}$, $\hat{x}_t = (\hat{x}_t)_{N_*}$, $J_t = (J_t)_{N_*, N_*}$, $K_t = (K_t)_{N_*, [1:n]}$, $P_{t|t-1} = (P_{t|t-1})_{N_*, N_*}$, $P_t = (P_t)_{N_*, N_*}$, and analogously for \tilde{x}_t , \tilde{J}_t , \tilde{K}_t , $\tilde{P}_{t|t-1}$, and \tilde{P}_t .

Note, however, that y_t and w_t may be supported on $[1 : n]$ and are thus not truncated when they appear; similarly, R is not truncated.

For $t > t_*$, both KFCS and GAKF run the same fixed-dimensional and fixed-parameter Kalman filter for $(x_t)_{N_*}$, but with different initial conditions. **Elaborate + possibly combine with first paragraph about DARE stuff below**

↓ ————— moved to enhance the flow of the proof

Notice that for $t \in [t_* : t_{**}]$ our model is a discrete-time time-invariant linear system with $(F, G, H, K) = (I, I, A, I)$. We can apply elements of linear system theory discussed in Section 5 to generate results relating to $P_{t|t-1}$.

ISSUE: our system doesn't match the usual $\mathbf{x} = \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{u}$, $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{K}\mathbf{u}$ model, so does this theory apply?

We see that

$$\begin{aligned} P_{t+1|t} &= P_t + Q \\ &= (I - K_t A) P_{t|t-1} + Q \\ &= P_{t|t-1} + Q - P_{t|t-1} A' (A P_{t|t-1} A' + R)^{-1} A P_{t|t-1}, \end{aligned}$$

which is a discrete algebraic Riccati recursion (DARR) with $F = I$, $G = I$, $Q = \sigma_{\text{sys}}^2 I_{|N_*| \times |N_*|}$, $R = \sigma_{\text{obs}}^2 I_{n \times n}$, and $S = 0$. **Verify Q, R.** Note that Q is constant on $[t_* : t_{**}]$ since we assume that \mathbf{D} occurs.

Since $|N_*| \leq S_{\max}$ and $\delta_{S_{\max}} < 1$, $A = (A_{[1:n], N_*})$ is full rank. Therefore, using the results from Section 5.1, $\{I, A\}$ is detectable. Further, since $Q^{1/2} = \sigma_{\text{sys}} I$ is full rank, $\{I, Q^{1/2}\}$ is stabilizable and unit-circle controllable.

From the algorithm, $P_{t|t-1} = P_{t-1} + Q_t \succeq 0$ for any t . **TRUE? (In theory yes, but not trivially true, need induction to show $P_t \succeq 0$ (hooray) or point out that P_t is a covariance, hence, PSD.) We need to be more careful here. Technically we are on $t \geq t_*$, may or may not affect things.**

Therefore, by [2], Lemma 8.7.3, the DARR converges to a positive semi-definite matrix P_* . This implies that $K_t \rightarrow K_* = P_* A' (A P_* A' + \sigma_{\text{obs}}^2 I_{n \times n})^{-1}$ and $J_t \rightarrow J_* = (I - K_* A)$. **VERIFY: $Q^{1/2}$ issue – see section on linear systems for explanation**

Further, since $\{I, A\}$ is detectable and $\{I, Q^{1/2}\}$ is unit-circle controllable, we can apply [3], Theorem E.5.1 to conclude that $\rho(I - K_* A) = \rho(J_*) < 1$.

Since GAKF and KFCS run the same Kalman filter, these results also apply to the GAKF iterates, i.e. $\tilde{P}_{t|t-1} \rightarrow P_*$, $\tilde{K}_t \rightarrow K_*$, and $\tilde{J}_t \rightarrow J_*$.

Define $\rho = \rho(J_*)$ and let $\varepsilon_0 = (1 - \rho)/2$. A standard result from linear algebra states that there exists a matrix norm $\|\cdot\|_\rho$ such that $\|J_*\|_\rho \leq \rho + \varepsilon_0 = (1 + \rho)/2 < 1$. Further, by the equivalence of matrix norms on a finite-dimensional space, there exists some constant $c_{2,\rho}$ such that $\|M\|_2 \leq c_{2,\rho} \|M\|_\rho$ for any matrix M .

Since $\tilde{J}_t \rightarrow J_*$, there exists some $t_c \geq t_0$ such that for all $t \geq t_c$, $\|\tilde{J}_t\|_2 < \|J_*\|_2 + 1$. Therefore, for any $t \geq t_0$, we have $\|\tilde{J}_t\|_2 \leq \max\{\|\tilde{J}_{t_0}\|_2, \|\tilde{J}_{t_0+1}\|_2, \dots, \|\tilde{J}_{t_c-1}\|_2, \|J_*\|_2 + 1\}$, i.e. there exists some value $\tilde{B}_J > 0$ such that $\|\tilde{J}_t\|_2 < \tilde{B}_J$ for all t . Since $\|\tilde{J}_t\|_2 < \infty$ for all t and $\|J_*\|_2 < \infty$, we must also have $\tilde{B}_J < \infty$.

By similar arguments, since J_t converges to J_* and $P_{t|t-1}$ and $\tilde{P}_{t|t-1}$ converge to P_* , there exist some $0 < B_J, B_P, \tilde{B}_P < \infty$ such that $\|J_t\|_2 < B_J$, $\|P_{t|t-1}\|_2 < B_P$, and $\|\tilde{P}_{t|t-1}\|_2 < \tilde{B}_P$ for all t .

Let $\varepsilon > 0$ be arbitrary.

The convergence results above and standard analysis techniques can be used to show that there exists some $t_\varepsilon > t_*$ such that for all $t \geq t_\varepsilon$, the following conditions hold:

- $\|K_t - \tilde{K}_t\|_2 < \varepsilon$;
- $\|J_t - \tilde{J}_t\|_2 < \varepsilon$; and
- $\|J_t\|_\rho \leq \|J_*\|_\rho + (1 - \rho)/4$.

★ Why do we care if t_ε is independent of y ? So that we can say t_{ms} is independent of y as well?

Problem: NV proof says \hat{x}_{t_*} is independent of y_{t_*} , but by definition it's not. AB draft: $t_* - 1$. So I agree that we're independent of $y_1 \dots y_{t_*-1}$, but we are dependent on $y_{t_*} \dots y_t$ because $\hat{x}_t = J_t \hat{x}_{t-1} + K_t y_t$ for $t > t_*$. **All of this independence stuff needs to be very carefully worked and verified; also, why do we care?**

↑———— /moved

Let $\hat{e}_t = x_t - \hat{x}_t$ and $\tilde{e}_t = x_t - \tilde{x}_t$. Define $\text{diff}_t = \hat{e}_t - \tilde{e}_t$ and notice that $\text{diff}_t = \tilde{x}_t - \hat{x}_t$.

Let $t > t_\varepsilon > t_*$. By the KFCS with LS algorithm and the model, we see that

$$\begin{aligned}
\hat{e}_t &= x_t - \hat{x}_t \\
&= (x_{t-1} + \nu_t) - (J_t \hat{x}_{t-1} + K_t y_t) \\
&= x_{t-1} + \nu_t - J_t \hat{x}_{t-1} - K_t (Ax_t + w_t) \\
&= x_{t-1} + \nu_t - J_t \hat{x}_{t-1} - K_t A(x_{t-1} + \nu_t) - K_t w_t \\
&= (I - K_t A)x_{t-1} - J_t \hat{x}_{t-1} + (I - K_t A)\nu_t - K_t w_t \\
&= J_t(x_{t-1} - \hat{x}_{t-1}) + J_t \nu_t - K_t w_t \\
&= J_t \hat{e}_{t-1} + J_t \nu_t - K_t w_t.
\end{aligned}$$

Similarly, using the GAKF algorithm and the model, we can verify that

$$\tilde{e}_t = \tilde{J}_t \tilde{e}_{t-1} + \tilde{J}_t \nu_t - \tilde{K}_t w_t.$$

Combining these results yields

$$\text{diff}_t = J_t \text{diff}_{t-1} + (J_t - \tilde{J}_t)(\tilde{e}_{t-1} + \nu_t) + (\tilde{K}_t - K_t)w_t.$$

Let

$$u_t = (J_t - \tilde{J}_t)(\tilde{e}_{t-1} + \nu_t) + (\tilde{K}_t - K_t)w_t,$$

so that $\text{diff}_t = J_t \text{diff}_{t-1} + u_t$. Recursively applying this identity, we see that

$$\begin{aligned}
\text{diff}_t &= J_t \text{diff}_{t-1} + u_t \\
&= J_t (J_{t-1} \text{diff}_{t-2} + u_{t-1}) + u_t \\
&= J_t J_{t-1} \text{diff}_{t-2} + J_t u_{t-1} + u_t \\
&= J_t J_{t-1} (J_{t-2} \text{diff}_{t-3} + u_{t-2}) + J_t u_{t-1} + u_t \\
&= J_t J_{t-1} J_{t-2} \text{diff}_{t-3} + J_t J_{t-1} u_{t-2} + J_t u_{t-1} + u_t \\
&\vdots \\
&= J_t J_{t-1} \cdots J_{t_\varepsilon+1} \text{diff}_{t_\varepsilon} + J_t J_{t-1} \cdots J_{t_\varepsilon+2} u_{t_\varepsilon+1} + \cdots + J_t u_{t-1} + u_t.
\end{aligned}$$

If we define

$$M_k^t = \begin{cases} J_t J_{t-1} \cdots J_{k+1} J_k & k \leq t \\ I & k > t \end{cases}$$

then we can more compactly write

$$\text{diff}_t = M_{t_\varepsilon+1}^t \text{diff}_{t_\varepsilon} + \sum_{k=t_\varepsilon+1}^t M_{k+1}^t u_k.$$

Therefore, applying the triangle and Cauchy-Schwarz inequalities for expectation and noting that the matrices $\{M_k^t\}$ are deterministic,

$$\begin{aligned}
\mathbb{E} [\|\text{diff}_t\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} &= \mathbb{E} \left[\left\| M_{t_\varepsilon+1}^t \text{diff}_{t_\varepsilon} + \sum_{k=t_\varepsilon+1}^t M_{k+1}^t u_k \right\|_2^2 \mid \mathbf{D} \right]^{\frac{1}{2}} \\
&\leq \mathbb{E} [\|M_{t_\varepsilon+1}^t \text{diff}_{t_\varepsilon}\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} + \mathbb{E} [\|u_t\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} + \sum_{k=t_\varepsilon+1}^{t-1} \mathbb{E} [\|M_{k+1}^t u_k\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} \\
&\leq \mathbb{E} [\|M_{t_\varepsilon+1}^t\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} \mathbb{E} [\|\text{diff}_{t_\varepsilon}\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} + \\
&\quad \mathbb{E} [\|u_t\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} + \sum_{k=t_\varepsilon+1}^{t-1} \mathbb{E} [\|M_{k+1}^t\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} \mathbb{E} [\|u_k\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} \\
\mathbb{E} [\|\text{diff}_t\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} &\leq \|M_{t_\varepsilon+1}^t\|_2 \mathbb{E} [\|\text{diff}_{t_\varepsilon}\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} + \\
&\quad \left(1 + \sum_{\ell=t_\varepsilon+2}^t \|M_\ell^t\|_2 \right) \max_{\tau \in [t_\varepsilon+1:t]} \left\{ \mathbb{E} [\|u_\tau\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} \right\}. \tag{2}
\end{aligned}$$

Recall that, for $k \geq t_\varepsilon$, we have

$$\|J_k\|_\rho \leq \|J_\star\|_\rho + (1 - \rho)/4 \leq (1 + \rho)/2 + (1 - \rho)/4 = (3 + \rho)/4 < 1.$$

Let $a = (3 + \rho)/4$. Then for $t_\varepsilon \leq k \leq t$,

$$\begin{aligned}
\|M_k^t\|_2 &\leq c_{2,\rho} \|M_k^t\|_\rho \\
&= \|J_t J_{t-1} \cdots J_k\|_\rho \\
&\leq \|J_t\|_\rho \|J_{t-1}\|_\rho \cdots \|J_k\|_\rho \\
\|M_k^t\|_2 &\leq c_{2,\rho} a^{t-k+1}. \tag{3}
\end{aligned}$$

With this, we see that

$$\begin{aligned}
\left(1 + \sum_{\ell=t_\varepsilon+2}^t \|M_\ell^t\|_2 \right) &\leq \left(1 + \sum_{\ell=t_\varepsilon+2}^t c_{2,\rho} a^{t-\ell+1} \right) \\
&\leq \max\{1, c_{2,\rho}\} \cdot \sum_{\ell=0}^{\infty} a^\ell \\
\left(1 + \sum_{\ell=t_\varepsilon+2}^t \|M_\ell^t\|_2 \right) &\leq \max\{1, c_{2,\rho}\} \cdot \frac{1}{1 - a}. \tag{4}
\end{aligned}$$

Let $\tau \in [t_\varepsilon + 1 : t]$ be arbitrary. Since $\tau > t_\varepsilon$, we have $\|\tilde{K}_\tau - K_\tau\|_2 < \varepsilon$ and $\|\tilde{J}_\tau - J_\tau\|_2 < \varepsilon$.

Consider

$$\begin{aligned}
\mathbb{E} [\|u_\tau\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} &= \mathbb{E} \left[\left\| (J_\tau - \tilde{J}_\tau)(\tilde{e}_{\tau-1} + \nu_\tau) + (\tilde{K}_\tau - K_\tau)w_\tau \right\|_2^2 \mid \mathbf{D} \right]^{\frac{1}{2}} \\
&\leq \|J_\tau - \tilde{J}_\tau\|_2 \mathbb{E} [\|\tilde{e}_{\tau-1} + \nu_\tau\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} + \|\tilde{K}_\tau - K_\tau\|_2 \mathbb{E} [\|w_\tau\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} \\
&< \varepsilon \cdot \mathbb{E} [\|\tilde{e}_{\tau-1} + \nu_\tau\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} + \varepsilon \cdot \mathbb{E} [\|w_\tau\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} \\
\mathbb{E} [\|u_\tau\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} &\leq \varepsilon \left(\mathbb{E} [\|\tilde{e}_{\tau-1}\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} + \mathbb{E} [\|\nu_\tau\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} + \mathbb{E} [\|w_\tau\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} \right), \tag{5}
\end{aligned}$$

where we have used the triangle and Cauchy-Schwarz inequalities for expectation.

By the properties of the Kalman filter, for any k , we have

$$\begin{aligned}
\tilde{P}_k &= \mathbb{E} [(\tilde{x}_k - \mathbb{E}[\tilde{x}_k \mid y_1, y_2, \dots, y_k])(\tilde{x}_k - \mathbb{E}[\tilde{x}_k \mid y_1, y_2, \dots, y_k])' \mid y_1, y_2, \dots, y_k] \\
&= \mathbb{E} [(\tilde{x}_k - x_k)(\tilde{x}_k - x_k)' \mid y_1, y_2, \dots, y_k] \\
&= \mathbb{E} [\tilde{e}_k \tilde{e}_k' \mid y_1, y_2, \dots, y_k] \\
&= \mathbb{E} [\tilde{e}_k \tilde{e}_k'],
\end{aligned}$$

where the independence on the last line follows because \tilde{P}_k has no dependence on any of the $\{y_i\}$, a well-known property of the Kalman filter (and consequence of the algorithm). Therefore,

$$\begin{aligned}
\mathbb{E} [\|\tilde{e}_k\|_2^2 \mid \mathbf{D}] &= \text{tr} (\mathbb{E} [\tilde{e}_k' \tilde{e}_k \mid \mathbf{D}]) \\
&= \mathbb{E} [\text{tr}(\tilde{e}_k' \tilde{e}_k) \mid \mathbf{D}] \\
&= \mathbb{E} [\text{tr}(\tilde{e}_k \tilde{e}_k') \mid \mathbf{D}] \\
&= \text{tr} (\mathbb{E} [\tilde{e}_k \tilde{e}_k' \mid \mathbf{D}]) \\
&= \text{tr} (\mathbb{E} [\tilde{e}_k \tilde{e}_k']) \\
&= \text{tr} (\tilde{P}_k).
\end{aligned}$$

Here, we used the fact that the occurrence of \mathbf{D} is independent of the value of $\tilde{P}_k = \mathbb{E} [e_k e_k']$. **Make sure this is legitimate.**

We see that

$$\|\tilde{P}_k\|_2 = \|\tilde{J}_k \tilde{P}_{k|k-1}\|_2 \leq \|\tilde{J}_k\|_2 \|\tilde{P}_{k|k-1}\|_2 < \tilde{B}_J \tilde{B}_P < \infty.$$

Since \tilde{P}_k is Hermitian, $\|\tilde{P}_k\|_2 = \lambda_{\max}(\tilde{P}_k)$. Therefore,

$$\text{tr}(\tilde{P}_k) = \sum_i \lambda_i(\tilde{P}_k) \leq |N_*| \lambda_{\max}(\tilde{P}_k) = |N_*| \|\tilde{P}_k\|_2 < |N_*| \tilde{B}_J \tilde{B}_P < \infty,$$

so there exists some $0 < \tilde{B} < \infty$ such that $\text{tr}(\tilde{P}_k) < \tilde{B}$ for all k .

Therefore,

$$\mathbb{E} [\|\tilde{e}_{\tau-1}\|_2^2 \mid \mathbf{D}] = \text{tr}(\tilde{P}_{\tau-1}) < \tilde{B}.$$

Since \mathbf{D} occurs, ν_τ is supported on N_* , so the covariance of $\nu_\tau = (\nu_\tau)_{N_*}$ is $\mathbb{E}[\nu_\tau \nu_\tau'] = \mathbb{E}[\nu_\tau \nu_\tau' \mid \mathbf{D}] = \sigma_{\text{sys}}^2 I_{|N_*| \times |N_*|}$. **VERIFY this claim.** Therefore,

$$\begin{aligned} \mathbb{E} [\|\nu_\tau\|_2^2 \mid \mathbf{D}] &= \text{tr}(\mathbb{E}[\nu_\tau' \nu_\tau \mid \mathbf{D}]) \\ &= \mathbb{E}[\text{tr}(\nu_\tau' \nu_\tau) \mid \mathbf{D}] \\ &= \mathbb{E}[\text{tr}(\nu_\tau \nu_\tau') \mid \mathbf{D}] \\ &= \text{tr}(\mathbb{E}[\nu_\tau \nu_\tau' \mid \mathbf{D}]) \\ &= \text{tr}(\sigma_{\text{sys}}^2 I_{|N_*| \times |N_*|}) \\ &= |N_*| \sigma_{\text{sys}}^2. \end{aligned}$$

A similar computation proves that $\mathbb{E} [\|w_\tau\|_2^2 \mid \mathbf{D}] = n \sigma_{\text{obs}}^2$.

With (5), these results show that

$$\mathbb{E} [\|u_\tau\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} < \varepsilon \left(\sqrt{\tilde{B}} + \sqrt{|N_*| \sigma_{\text{sys}}^2} + \sqrt{n \sigma_{\text{obs}}^2} \right).$$

Since $\tau \in [t_\varepsilon + 1 : t]$ was arbitrary, we conclude that

$$\max_{\tau \in [t_\varepsilon + 1 : t]} \left\{ \mathbb{E} [\|u_\tau\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} \right\} < \varepsilon \left(\sqrt{\tilde{B}} + \sqrt{|N_*| \sigma_{\text{sys}}^2} + \sqrt{n \sigma_{\text{obs}}^2} \right). \quad (6)$$

We have seen that $\mathbb{E} [\|\tilde{e}_k\|_2^2 \mid \mathbf{D}] < \text{tr}(\tilde{P}_k) < \tilde{B}$ for some \tilde{B} and all k ; by similar work, we can conclude that there exists some B such that $\mathbb{E} [\|e_k\|_2^2 \mid \mathbf{D}] < \text{tr}(P_k) < B$ for all k . Therefore, by the triangle inequality for expectation,

$$\begin{aligned} \mathbb{E} [\|\text{diff}_{t_\varepsilon}\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} &= \mathbb{E} [\|e_{t_\varepsilon} - \tilde{e}_{t_\varepsilon}\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} \\ &\leq \mathbb{E} [\|e_{t_\varepsilon}\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} + \mathbb{E} [\|\tilde{e}_{t_\varepsilon}\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} \\ \mathbb{E} [\|\text{diff}_{t_\varepsilon}\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} &< B + \tilde{B}. \end{aligned} \quad (7)$$

Combining (2) with (3), (4), (6), and (7), we see that

$$\mathbb{E} [\|\text{diff}_t\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} < c_{2,\rho} a^{t-t_\varepsilon} (B + \tilde{B}) + C\varepsilon,$$

where $C = \max\{1, c_{2,\rho}\} \cdot \frac{1}{1-a} \cdot \left(\sqrt{\tilde{B}} + \sqrt{|N_*| \sigma_{\text{sys}}^2} + \sqrt{n \sigma_{\text{obs}}^2} \right)$.

If

$$t_{\text{ms}} = t_{\varepsilon} + \left\lceil \log_a \left(\frac{C\varepsilon}{c_{2,\rho}(B + \tilde{B})} \right) \right\rceil,$$

then we see that for all $t \geq t_{\text{ms}}$,

$$\mathbb{E} [\|\tilde{x}_t - \hat{x}_t\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} = \mathbb{E} [\|\text{diff}_t\|_2^2 \mid \mathbf{D}]^{\frac{1}{2}} < 2C\varepsilon,$$

and since C is constant and ε is arbitrary we have obtained our desired result. □

Corollary 1. *Assume that the conditions of Lemma 2 hold.*

Then given any ε and ε_{err} there exists some $\tau_{KF} = \tau_{KF}(\varepsilon, \varepsilon_{\text{err}}, N_)$ such that for all $t \in [t_* + \tau_{KF} : t_{**}]$, we have $\Pr(\|\tilde{x}_t - \hat{x}_t\|_2^2 \leq \varepsilon_{\text{err}} \mid \mathbf{D}) > 1 - \varepsilon$. Note that if $t_* + \tau_{KF} > t_{**}$, then this interval is empty and the result is vacuously true.*

Proof. Let $\varepsilon > 0$ and $\varepsilon_{\text{err}} > 0$ be given and let $\tilde{\varepsilon} \leq \varepsilon \cdot \varepsilon_{\text{err}}$. By Lemma 2, there exists some $t_{\text{ms}} = t_{\text{ms}}(\tilde{\varepsilon}, N_*)$ such that for all $t \geq t_{\text{ms}}$,

$$\mathbb{E}[\|\tilde{x}_t - \hat{x}_t\|_2^2 \mid \mathbf{D}] < \tilde{\varepsilon} \leq \varepsilon \cdot \varepsilon_{\text{err}}.$$

Let $t \geq t_{\text{ms}}$. By Markov's inequality,

$$\Pr(\|\tilde{x}_t - \hat{x}_t\|_2^2 > \varepsilon_{\text{err}} \mid \mathbf{D}) \leq \frac{\mathbb{E}[\|\tilde{x}_t - \hat{x}_t\|_2^2 \mid \mathbf{D}]}{\varepsilon_{\text{err}}} < \frac{\tilde{\varepsilon}}{\varepsilon_{\text{err}}} \leq \varepsilon.$$

Define $\tau_{KF} = t_{\text{ms}} - t_*$. Since t_{ms} is a function of $\tilde{\varepsilon}$, which is itself a function of ε and ε_{err} , we have $\tau_{KF} = \tau_{KF}(\varepsilon, \varepsilon_{\text{err}}, N_*)$, and for all $t \geq t_{\text{ms}} = t_* + \tau_{KF}$,

$$\Pr(\|\tilde{x}_t - \hat{x}_t\|_2^2 < \varepsilon_{\text{err}} \mid \mathbf{D}) > 1 - \varepsilon,$$

which is our desired result. □

Theorem 1. Assume that the conditions of Lemma 1 and Lemma 2 hold.

Choose d **large enough**.

- 1) Given any $j \in [0 : K-1]$, $\Pr(\|\tilde{x}_t - \hat{x}_t\|_2^2 \leq \varepsilon_{err}) > (1-\varepsilon)^{j+2}$ for all $t \in [t_j + \tau_{det} + \tau_{KF} : t_{j+1} - 1]$.
- 2) $\Pr(|\Delta_t| \leq S_{add} \text{ and } |\Delta_e| = 0 \text{ for all } t \geq t_0) \geq (1-\varepsilon)^K$
- 3) $\Pr(\text{For all } j \in [0 : K-1], |\Delta_t| = 0 \text{ and } |\Delta_e| = 0 \text{ for all } t \in [t_j + \tau_{det} : t_{j+1} - 1]) \geq (1-\varepsilon)^K$

Proof. We first show by induction that $\Pr(\mathbf{E}_j) \geq (1-\varepsilon)^{j+1}$ for all $j \geq 0$.

Consider the base case, where $j = 0$. We have

$$\Pr(\mathbf{E}_0) = \Pr(\mathbf{E}_0 | \mathbf{F}_0) \geq 1 - \varepsilon$$

by Lemma 1, which proves the base case. **Go back and add in something about j to statement of Lemma 1, also handle the case with E_{-1} / explain the case with E_0**

Now assume that the claim is true for $j = (k-1)$ for some $k \geq 1$, that is, $\Pr(\mathbf{E}_{k-1}) \geq (1-\varepsilon)^k$. Consider

$$\begin{aligned} \Pr(\mathbf{E}_k) &= \Pr(\mathbf{E}_k \cap \mathbf{E}_{k-1}) + \Pr(\mathbf{E}_k \cap (\mathbf{E}_{k-1})^c) \\ &\geq \Pr(\mathbf{E}_k \cap \mathbf{E}_{k-1}) \\ &= \Pr(\mathbf{E}_k | \mathbf{E}_{k-1}) \Pr(\mathbf{E}_{k-1}) \\ &= \Pr(\mathbf{E}_k | \mathbf{F}_k) \Pr(\mathbf{E}_{k-1}) \quad \text{WHY is this true?} \\ &\geq (1-\varepsilon)(1-\varepsilon)^k \\ &= (1-\varepsilon)^{k+1}, \end{aligned}$$

where we applied Lemma 1 to conclude that $\Pr(\mathbf{E}_k | \mathbf{F}_k) \geq 1-\varepsilon$. Therefore, by the principle of mathematical induction, we conclude that

$$\Pr(\mathbf{E}_j) \geq (1-\varepsilon)^{j+1}$$

for all $j \geq 0$.

Fix $j \in [0 : K-1]$.

(needed?) Fix $t \in [t_j + \tau_{det} + \tau_{KF} : t_{j+1} - 1]$.

Choosing $t_* = t_j + \tau_{det}$ and $t_{**} = t_{j+1} - 1$, the event $\mathbf{D} = \{\hat{N}_t = N_t = N_* \text{ for all } t \in [t_* : t_{**}]\}$ is identically the event $\mathbf{E}_j = \{\hat{N}_t = N_t \text{ for all } t \in [t_j + \tau_{det} : t_{j+1} - 1]\}$. Corollary 1 thus yields

$$\Pr(\|\tilde{x}_t - \hat{x}_t\|_2^2 \leq \varepsilon_{err} | \mathbf{E}_j) > 1 - \varepsilon$$

for all $t \in [t_j + \tau_{\text{det}} + \tau_{\text{KF}} : t_{j+1} - 1]$.

Note that since $d > \tau_{\text{det}} + \tau_{\text{KF}}$ for all j , the interval $[t_j + \tau_{\text{det}} + \tau_{\text{KF}} : t_{j+1} - 1]$ is nonempty.

For any $t \in [t_j + \tau_{\text{det}} + \tau_{\text{KF}} : t_{j+1} - 1]$, we see that

$$\begin{aligned}
\Pr(\|\tilde{x}_t - \hat{x}_t\|_2^2 \leq \varepsilon_{\text{err}}) &= \Pr(\{\|\tilde{x}_t - \hat{x}_t\|_2^2 \leq \varepsilon_{\text{err}}\} \cap \mathbf{E}_j) + \Pr(\{\|\tilde{x}_t - \hat{x}_t\|_2^2 \leq \varepsilon_{\text{err}}\} \cap (\mathbf{E}_j)^c) \\
&\geq \Pr(\{\|\tilde{x}_t - \hat{x}_t\|_2^2 \leq \varepsilon_{\text{err}}\} \cap \mathbf{E}_j) \\
&= \Pr(\|\tilde{x}_t - \hat{x}_t\|_2^2 \leq \varepsilon_{\text{err}} \mid \mathbf{E}_j) \Pr(\mathbf{E}_j) \\
&> (1 - \varepsilon)(1 - \varepsilon)^{j+1} \\
&= (1 - \varepsilon)^{j+2},
\end{aligned}$$

which verifies **the first claim**.

I think that the third claim's probability equals the one below. Either way, we need this.

$$\begin{aligned}
\Pr(\mathbf{E}_0 \cap \mathbf{E}_1 \cap \dots \cap \mathbf{E}_{K-1}) &= \Pr(\mathbf{E}_0) \Pr(\mathbf{E}_1 \mid \mathbf{E}_0) \Pr(\mathbf{E}_2 \mid \mathbf{E}_0 \cap \mathbf{E}_1) \dots \Pr\left(\mathbf{E}_{K-1} \mid \bigcap_{j=0}^{K-1} \mathbf{E}_j\right) \\
&= \Pr(\mathbf{E}_0) \Pr(\mathbf{E}_1 \mid \mathbf{E}_0) \Pr(\mathbf{E}_2 \mid \mathbf{E}_1) \dots \Pr(\mathbf{E}_{K-1} \mid \mathbf{E}_{K-2}) \\
&= \Pr(\mathbf{E}_0) \Pr(\mathbf{E}_1 \mid \mathbf{F}_1) \Pr(\mathbf{E}_2 \mid \mathbf{F}_2) \dots \Pr(\mathbf{E}_{K-1} \mid \mathbf{F}_{K-1}) \\
&\geq (1 - \varepsilon)^K
\end{aligned}$$

Stuff to verify:

$\Pr(E_j \mid E_{j-1}) = \Pr(E_j \mid F_j)$: if E_{j-1} happens, then F_j definitely happens, but not seeing why these are equal yet.

Markov property used on $\{E_j\}$: justification

□

References

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