KF-CS Theory

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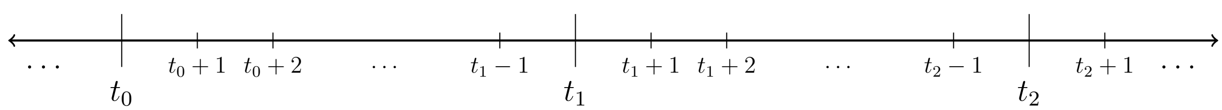
1 Model

At each time t, we have $y_t = Ax_t + w_t$.

Model on $\{x_t\}$.

Time indices are discrete. Make the distinction between sampling times (used) and continuous time (not used).



At the addition times $t_j = t_0 + jd$ for some t_0, the support of x_t changes: $N_t = N_{t_j}$ for all $t \in [t_j : t_{j+1} - 1]$, and $N_{t_j} \subset N_{t_{j+1}}$.

2 Algorithm – KF-CS with LS

(TBA)

3 Candes RIP – C_1 Computation for α

We need to add this as a theorem or something – cite [1] Thm 1.3 and explicitly give the value of C_1 and the commentary below.

THEOREM / RESULT: [1], Theorem 1.3

Suppose $y = Ax + \eta$, $|\operatorname{supp}(x)| = s$, $\delta_{2s} = \delta_{2s}(A) < \sqrt{2} - 1$, and $\|\eta\|_2 \leq \xi$. Then

$$\hat{x} = \arg\min_z \|z\|_1 \text{ subject to } \|y - Az\|_2 \leq \xi$$

satisfies

$$\|x - \hat{x}\|_2 \leq C_1(s)\xi,$$

where

$$C_1(s) = \frac{4\sqrt{1 + \delta_{2s}}}{1 - (1 + \sqrt{2})\delta_{2s}} \ .$$

Claim / Note: It can be shown that C_1 is an increasing function of δ_{2s}, and δ_{2s} is an increasing function of s, so C_1 is an increasing function of s.

For any support size S in this paper, we will have $S \leq S_{\max}$ and thus $C_1(S) \leq C_1(S_{\max})$.

4 Proofs

Lemma 1. *Assume that $\{x_t\}$ follow the signal model above, $y_t = Ax_t + w_t$, $\{t_0, t_0+1, t_0+2, \ldots\}$ is a discrete set of sampling times, only additions to true support ($N_t \subseteq N_{t+1}$ for all t), etc.*

Further assume that

i) The true solution is exactly recovered at the initial time t_0: $\hat{x}_{t_0} = x_{t_0}$, so $\hat{N}_{t_0} = N_{t_0} = N_0$; ***Can we relax this to just the true support is recovered?***

ii) The maximum support size S_{max} satisfies $S_{max} \leq S_{**} = \max\{s : \delta_{2s}(A) < \sqrt{2} - 1\}$;

iii) The observation noise w_t is bounded in magnitude: $\|w_t\| < \xi$ for all t and some $\xi > 0$;

iv) The addition thresholds α_t satisfy $\alpha_t = \alpha = C\xi$ for all t, where

$$C = C(S_{max}) = \frac{4\sqrt{1 + \delta_{2S_{max}}}}{1 - \left(1 + \sqrt{2}\right)\delta_{2S_{max}}}$$

with $\delta_{2S_{max}} = \delta_{2S_{max}}(A)$; and

v) The addition delay d satisfies $d > \tau_{det}$, where the detection delay τ_{det} is defined by

$$\tau_{det} = \tau_{det}(\alpha, \varepsilon) = \left\lceil \left(\frac{2\alpha}{\sigma_{sys} \mathcal{Q}^{-1}\left(\frac{(1-\varepsilon)^{1/S_{add}}}{2}\right)} \right)^2 \right\rceil .$$

Here, $\mathcal{Q}^{-1}(x)$ is the inverse of the Gaussian $\mathcal{Q}$-function, $\mathcal{Q}(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt$.

Then

1) $\|x_t - \hat{x}_{t,CSres}\|_2 \leq \alpha$ for all sampling times t;

2) there are no false support additions: $\hat{N}_t \subseteq N_t$ for all sampling times t; and

3) $\boldsymbol{Pr}(E_j \mid F_j) \geq 1 - \varepsilon$, where $E_j = \{\hat{N}_t = N_t \text{ for all } t \in [t_j + \tau_{det} : t_{j+1} - 1]\}$ and $F_j = \{\hat{N}_{t_j - 1} = N_{t_j - 1}\}$.

We may want to split claim 3 into its own piece because its proof relies on the other 2 parts, which are proved separately with induction.

Proof. **Need to find some way to get Candes Thm 1.3 in here and make the connection that $\hat{x}_{t,\mathbf{CSres}}$ in our notation is $x^\star$ in his. Also need to point out that the way we chose α, we have any $C_1\xi \leq C_1(S_{\mathbf{max}})\xi = \alpha$.**

To prove claims 1 and 2, we proceed by induction on the value of t.

Consider the base case, where $t = t_0$. Claim 1 follows from Theorem 1.3 in [1] and assumptions (ii), (iii), and (iv) **(Not immediate – need to connect to Candes as above)**, and assumption (i) trivially proves claim 2.

Suppose now that claims 1 and 2 are both true for some time $(t-1)$. We show that the claims are true at time t.

First, we verify claim 1 at time t. Let

$$\begin{aligned}\beta_t &= x_t - \hat{x}_{t,\text{init}} \\ \hat{\beta}_t &= \arg\min_{\beta} \|\beta\|_1 \text{ subject to } \|y_t - A\beta\|_2 < \xi \\ \hat{x}_{t,\text{CSres}} &= \hat{x}_{t,\text{init}} + \hat{\beta}_t,\end{aligned}$$

where $\hat{x}_{t,\text{init}}$ is defined in the algorithm and satisfies $\text{supp}(\hat{x}_{t,\text{init}}) = \hat{N}_{t-1}$.

By the induction hypothesis, $\hat{N}_{t-1} \subseteq N_{t-1}$, and by our model assumptions we have $N_{t-1} \subseteq N_t$. Therefore, $\text{supp}(\beta_t) \subseteq N_t \cup N_{t-1} = N_t$, so $|\text{supp}(\beta_t)| \leq |N_t| \leq S_{\max}$. With this, we can apply Theorem 1.3 in [1] to see that $\|\beta_t - \hat{\beta}_t\|_2 \leq \alpha$ **(AGAIN, need to make this connection)**. By the definitions of β_t and $\hat{x}_{t,\text{CSres}}$, we see that $\|\beta_t - \hat{\beta}_t\|_2 = \|x_t - \hat{x}_{t,\text{CSres}}\|_2$, so claim 1 follows.

Next, we verify claim 2 at time t. Suppose that $(x_t)_i = 0$ for some index i, so that $i \notin \text{supp}(x_t) = N_t$. Since $N_{t-1} \subseteq N_t$, we must also have $i \notin N_{t-1}$; by the induction hypothesis, this implies that $i \notin \hat{N}_{t-1}$.

Applying the result of claim 1,

$$|(\hat{x}_{t,\text{CSres}})_i| = |(x_t - \hat{x}_{t,\text{CSres}})_i| \leq \|x_t - \hat{x}_{t,\text{CSres}}\|_2 \leq \alpha.$$

Referring to the algorithm, $\hat{N}_t = \hat{N}_{t-1} \cup \{j : |(\hat{x}_{t,\text{CSres}})_j| > \alpha\}$. Since $i \notin \hat{N}_{t-1}$ and $|(\hat{x}_{t,\text{CSres}})_i| \leq \alpha$, it follows that $i \notin \hat{N}_t$. Thus if $i \notin N_t$, then $i \notin \hat{N}_t$; equivalently, if $i \in \hat{N}_t$, then $i \in N_t$. Therefore, $\hat{N}_t \subseteq N_t$, which proves claim 2 and completes our induction proof.

Now, we prove claim 3. Let $\Delta_t = N_t \setminus \hat{N}_{t-1}$ denote the set of indices of the true support at time t which have not been detected before time t. Suppose that F_j holds, that is, $\hat{N}_{t_j-1} = N_{t_j-1}$.

Since F_j holds, $\Delta_t \subseteq \Delta_{\text{add},t_j}$ for all $t \in [t_j : t_{j+1}-1]$.

Let $i \in \Delta_t$ for some $t \in [t_j : t_{j+1}-1]$ and suppose that $|(x_t)_i| > 2\alpha$. Applying the result from claim 1,

$$0 \leq |(x_t - \hat{x}_{t,\text{CSres}})_i| \leq \|(x_t - \hat{x}_{t,\text{CSres}})\|_2 \leq \alpha < 2\alpha < |(x_t)_i|,$$

so that

$$\begin{aligned}
|(\hat{x}_{t,\text{CSres}})_i| &= |(x_t)_i - [(x_t)_i - (\hat{x}_{t,\text{CSres}})_i]| \\
&\geq \big||(x_t)_i| - |(x_t - \hat{x}_{t,\text{CSres}})_i|\big| \\
&= |(x_t)_i| - |(x_t - \hat{x}_{t,\text{CSres}})_i| \\
&> 2\alpha - \alpha \\
&= \alpha.
\end{aligned}$$

We see that if $|(x_t)_i| > 2\alpha$, then $|(\hat{x}_{t,\text{CSres}})_i| > \alpha$, so $i \in \hat{N}_t = \hat{N}_{t-1} \cup \{j : |(\hat{x}_{t,\text{CSres}})_j| > \alpha\}$.

If $|(x_t)_i| > 2\alpha$ for all $i \in \Delta_{\text{add},t_j}$, then $\Delta_t \subseteq \Delta_{\text{add},t_j} \subseteq \hat{N}_t$; in words, we will detect all "missing" indices at time t, so $\hat{N}_t = N_t$.

From the above discussion, we see that the event $\{|(x_t)_i| > 2\alpha \text{ for all } i \in \Delta_{\text{add},t_j}\}$ is contained within the event $\{|(x_t)_i| > 2\alpha \text{ for all } i \in \Delta_t \mid F_j\}$, which in turn is contained within the event $\{\hat{N}_t = N_t \mid F_j\}$.

All of the above is still kind of weak in places. It all makes sense in words and is true, but the math / set theory is kind of wonky.

Our model asserts that the entries $(x_t)_i$ of x_t are independent and identically distributed $\mathcal{N}\left(0, (t-t_j)\sigma_{\text{sys}}^2\right)$ random variables. With this in mind, we see that

$$\begin{aligned}
\mathbf{Pr}(\hat{N}_t = N_t \mid F_j) &\geq \mathbf{Pr}\left(|(x_t)_i| > 2\alpha \text{ for all } i \in \Delta_t \mid F_j\right) \\
&\geq \mathbf{Pr}\left(|(x_t)_i| > 2\alpha \text{ for all } i \in \Delta_{\text{add},t_j}\right) \\
&= \left[\mathbf{Pr}\left(|(x_t)_1| > 2\alpha\right)\right]^{S_{\text{add}}} \\
&= \left[2\mathcal{Q}\left(\frac{2\alpha}{\sigma_{\text{sys}}\sqrt{t-t_j}}\right)\right]^{S_{\text{add}}}.
\end{aligned}$$

We examine the particular case there $t = t_j + \tau_{\text{det}}$. In this case,

$$\begin{aligned}
\mathbf{Pr}(\hat{N}_{t_j+\tau_{\text{det}}} = N_{t_j+\tau_{\text{det}}} \mid F_j) &\geq \left[2\mathcal{Q}\left(\frac{2\alpha}{\sigma_{\text{sys}}\sqrt{(t_j+\tau_{\text{det}})-t_j}}\right)\right]^{S_{\text{add}}} \\
&= \left[2\mathcal{Q}\left(\frac{2\alpha}{\sigma_{\text{sys}}\sqrt{\tau_{\text{det}}}}\right)\right]^{S_{\text{add}}} \\
&\geq 1-\varepsilon,
\end{aligned}$$

where the final inequality is easily verified and follows from the ceiling in the definition of τ_{det} and the fact that $\mathcal{Q}$ is a decreasing function.

If $\hat{N}_t = N_t$ for $t = t_j + \tau_{\text{det}}$, then the model assumptions of no support deletions and no support additions until time t_{j+1}, in addition to the result of claim 2, imply that $\hat{N}_t = N_t$ for all $t \in [t_j + \tau_{\text{det}} : t_{j+1} - 1]$, which is exactly the event E_j. Therefore, $\mathbf{Pr}(E_j \mid F_j) = \mathbf{Pr}(\hat{N}_{t_j+\tau_{\text{det}}} = N_{t_j+\tau_{\text{det}}} \mid F_j) \geq 1-\varepsilon$, which completes the proof. $\square$

References

[1] Emmanuel J. Candès, *The restricted isometry property and its implications for compressed sensing*, Comptes Rendus Mathematique, Volume 346, Issues 9–10, May 2008, Pages 589-592, ISSN 1631-073X, http://dx.doi.org/10.1016/j.crma.2008.03.014.