

KFCS Theory

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Last compiled July 23, 2014

1 Model

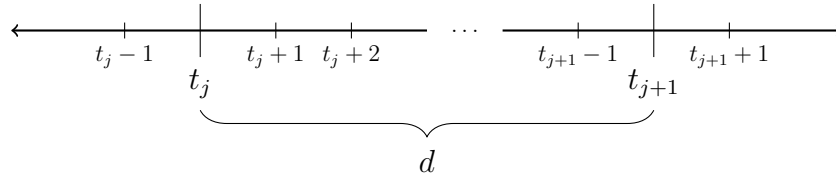
At each time $t \geq t_0$, we have

$$\begin{aligned} y_t &= Ax_t + w_t \\ x_{t+1} &= x_t + \nu_{t+1} \end{aligned} \quad \text{Can we use } \nu_t?$$

Here, $\mathbb{E}[w_t] = \mathbf{0}$, $\text{cov}(w_t) = \mathbb{E}[w_t w_t'] = \sigma_{\text{obs}}^2 I_n$, iid and independent of x_t ; $x_{t_0} = x_0 \sim \mathcal{N}(\mathbf{0}, \sigma_{\text{sys},0}^2 I_{N_0})$; and $\nu_t \sim \mathcal{N}(\mathbf{0}, \sigma_{\text{sys}}^2 I_{N_t})$ iid.

$y_t, w_t \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times m}$, $x_t, \nu_t \in \mathbb{R}^m$.

Time indices are discrete. Make the distinction between sampling times (used) and continuous time (not used).



At the addition times $t_j = t_0 + jd$ for some t_0 , the support of x_t changes: $N_t = N_{t_j}$ for all $t \in [t_j : t_{j+1} - 1]$, and $N_{t_j} \subset N_{t_{j+1}}$.

2 Algorithm – KFCS with LS

This algorithm applies to the case where there are no support deletions.

Issues:

P_{t_0-1} and Q_t – is this an identity of size $|N_{hat}|$ or is it a full-blown identity with nonzeros on diagonal entries corresponding to N_{hat}

Is this algorithm transcribed correctly? There are 3 versions of it that I have (NV original, AB+NV typed draft, and AB handwritten) and all 3 are different.

Look for places to simplify – this is long and contains repeat steps, which is non-ideal

Input: $\sigma_{\text{sys}}, \sigma_{\text{obs}}, \sigma_{\text{sys},0}, A, \{t_j\}, \{N_t\}, \{y_t\}$

$$\hat{x}_{t_0, \text{init}} = \arg \min_x \|x\|_1 \text{ subject to } \|y_{t_0} - Ax\|_2 < \xi$$

$$\hat{N}_{t_0} = \{j : |(\hat{x}_{t_0, \text{init}})_j| > \alpha\}$$

$$P_{t_0-1} = \sigma_{\text{sys},0}^2 I_{\hat{N}_{t_0}}$$

$$Q_{t_0} = 0$$

$$\hat{x}_{t_0-1} = \mathbf{0}$$

$$P_{t_0|t_0-1} = P_{t_0-1} + Q_{t_0}$$

$$K_{t_0} = P_{t_0|t_0-1} A' (A P_{t_0|t_0-1} A' + \sigma_{\text{obs}}^2 I)^{-1}$$

$$J_{t_0} = I - K_{t_0} A$$

$$P_{t_0} = J_{t_0} P_{t_0|t_0-1}$$

$$\hat{x}_{t_0} = J_{t_0} \hat{x}_{t_0-1} + K_{t_0} y_{t_0}$$

for $t > t_0$ **do**

$$Q_t = \sigma_{\text{sys}}^2 I_{\hat{N}_{t-1}}$$

$$P_{t|t-1} = P_{t-1} + Q_t$$

$$K_t = P_{t|t-1} A' (A P_{t|t-1} A' + \sigma_{\text{obs}}^2 I)^{-1}$$

$$J_t = I - K_t A$$

$$P_t = J_t P_{t|t-1}$$

$$\hat{x}_{t, \text{init}} = J_t \hat{x}_{t-1} + K_t y_t$$

$$y_{t, \text{res}} = y_t - A \hat{x}_{t, \text{init}}$$

$$\hat{\beta}_t = \arg \min_{\beta} \|\beta\|_1 \text{ subject to } \|y_{t, \text{res}} - A\beta\|_2 < \xi$$

$$\hat{x}_{t, \text{CSres}} = \hat{x}_{t, \text{init}} + \hat{\beta}_t$$

$$\Delta_A = \{j : |(\hat{x}_{t, \text{CSres}})_j| > \alpha\}$$

$$\hat{N}_t = \hat{N}_{t-1} \cup \Delta_A$$

if $\Delta_A = \emptyset$ **then**

$$| \hat{x}_t = \hat{x}_{t, \text{init}}$$

else

$$| \hat{x}_t = \mathbf{0}$$

$$| (\hat{x}_t)_{\hat{N}_t} = (A_{[1:n], \hat{N}_t})^\dagger y_t$$

$$| P_t = 0_{m \times m}$$

$$| (P_t)_{\hat{N}_t, \hat{N}_t} = \left[(A_{[1:n], \hat{N}_t})' (A_{[1:n], \hat{N}_t}) \right]^{-1} \sigma_{\text{obs}}^2 I_{|\hat{N}_t|}$$

end

end

3 Algorithm – Genie-Aided Kalman Filtering (GAKF)

This algorithm applies to the case where there are no support deletions.

Issues:

Check blue piece below – do we want all-ones, identity of size $|\Delta A|$, or identity restricted to ΔA and zero else?

Input: $\sigma_{\text{sys}}, \sigma_{\text{obs}}, \sigma_{\text{sys},0}, A, \{t_j\}, \{N_t\}, \{y_t\}$

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for  $t \geq t_0$  do
  if  $t = t_0$  then
     $T = N_0$ 
     $\tilde{P}_{t-1} = \sigma_{\text{sys},0}^2 I_T$ 
     $\tilde{x}_{t-1} = \mathbf{0}$ 
     $\tilde{Q}_t = 0$ 
  else
     $T = N_{t-1}$ 
     $\tilde{Q}_t = \sigma_{\text{sys}}^2 I_T$ 
    if  $t = t_j$  for some  $j > 0$  then
       $\Delta_A = N_t \setminus N_{t-1}$ 
       $\left( \tilde{P}_{t-1} \right)_{\Delta_A, \Delta_A} = \sigma_{\text{sys}}^2 I_{|\Delta_A|}$ 
    end
  end
   $\tilde{P}_{t|t-1} = \tilde{P}_{t-1} + \tilde{Q}_t$ 
   $\tilde{K}_t = \tilde{P}_{t|t-1} A' \left( A \tilde{P}_{t|t-1} A' + \sigma_{\text{obs}}^2 I \right)^{-1}$ 
   $\tilde{J}_t = I - \tilde{K}_t A$ 
   $\tilde{P}_t = \tilde{J}_t \tilde{P}_{t|t-1}$ 
   $\tilde{x}_t = \tilde{J}_t \tilde{x}_{t-1} + \tilde{K}_t y_t$ 
end

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4 Candes RIP – C_1 Computation for α

We need to add this as a theorem or something – cite [1] Thm 1.3 and explicitly give the value of C_1 and the commentary below.

THEOREM / RESULT: [1], Theorem 1.3

Suppose $y = Ax + \eta$, $|\text{supp}(x)| = s$, $\delta_{2s} = \delta_{2s}(A) < \sqrt{2} - 1$, and $\|\eta\|_2 \leq \xi$. Then

$$\hat{x} = \arg \min_z \|z\|_1 \text{ subject to } \|y - Az\|_2 \leq \xi$$

satisfies

$$\|x - \hat{x}\|_2 \leq C_1(s)\xi,$$

where

$$C_1(s) = \frac{4\sqrt{1 + \delta_{2s}}}{1 - (1 + \sqrt{2})\delta_{2s}}.$$

Claim / Note: It can be shown that C_1 is an increasing function of δ_{2s} , and δ_{2s} is an increasing function of s , so C_1 is an increasing function of s .

For any support size S in this paper, we will have $S \leq S_{\max}$ and thus $C_1(S) \leq C_1(S_{\max})$.

5 Kailath, Sayed, Hassibi – Linear Estimation

5.1 Appendix C, Section 3

Definitions and concepts from system theory.

$$\begin{aligned}x_{i+1} &= F_i x_i + G_i u_i \\ y_i &= H_i x_i + K_i u_i\end{aligned}$$

Shorthand (F_i, G_i, H_i, K_i) . **PROBLEM:** y is tied also to u here instead of some other variable...

Stable: F is stable if $|\lambda_i| < 1$ for all $\lambda_i \in \sigma_p(F)$, equivalently, $\rho(F) < 1$.

Controllable: $\{F, G\}$ is controllable if and only if the controllability matrix $\mathcal{C} = [G, FG, F^2G, \dots, F^{n-1}G]$ is full rank n .

Unit-circle Controllable: $\{F, G\}$ is unit-circle controllable if $\text{rank}(\lambda I - FG) = n$ at all unit-circle eigenvalues λ of F .

P. 502 – unit-circle controllable \equiv there exists K with $F - GK$ has no unit-circle eigenvalues. Is this even true? Nontrivial if so.

Stabilizable: $\{F, G\}$ is stabilizable if and only if $\text{rank}(\lambda I - FG) = n$ for all $\lambda \in \sigma_p(F)$ with $|\lambda| \geq 1$.

Observable: $\{F, H\}$ is observable if and only if $\{F^*, H^*\}$ is controllable, i.e. $\mathcal{C} = [H^*, F^*H^*, (F^*)^2H^*, \dots, (F^*)^{n-1}H^*]$ is full rank n .

Detectable: $\{F, H\}$ is detectable if and only if $\{F^*, H^*\}$ is stabilizable, i.e. $\text{rank}(\lambda I - F^*H^*) = n$ for all $\lambda \in \sigma_p(F)$ with $|\lambda| \geq 1$.

5.2 Chapter 8, section 3

Solutions of a DARE

Section 8.1 – Time-Invariant State-Space Model (p.266)

$$\begin{aligned}x_{i+1} &= Fx_i + Gu_i \\ y_i &= Hx_i + v_i \\ Q &= \text{cov}(u), \quad R = \text{cov}(v), \quad S = \text{cov}(u, v)\end{aligned}$$

Discrete Algebraic Riccati Equation (DARE):

$$P = FPF^* + GQG^* - K_p R_e K_p^*$$

$$R_e = R + HPH^*$$

$$K_p = (FPH^* + GS)R_e^{-1}$$

Define $F^s = F - GSR^{-1}H$ and $Q^s = Q - SR^{-1}S^*$.

Theorem 8.3.1: Assume that F is stable (otherwise, $\{F, H\}$ is detectable), $\{F^s, GQ^{s/2}\}$ controllable on the unit circle,

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \succcurlyeq 0, \quad R \succ 0.$$

Under these conditions, the DARE has a unique solution P such that $F - K_pH$ is stable. Moreover, this so-called stabilizing solution is positive semi-definite and results in a positive definite $R_e = R + HPH^*$.

5.3 Chapter 14

Lemma 14.2.1: Consider the zero-initial-condition Riccati recursion and assume $\{F, H\}$ detectable and $\{F, GQ^{1/2}\}$ unit-circle controllable so that the unique stabilizing solution P exists. Then P_i^0 converges to P if and only if $\{F, GQ^{1/2}\}$ is stabilizable.

I don't think we're in the zero-initial-condition case; rather, $P^0 = \sigma I$.

Exercise 14.4? p.546

5.4 Appendix E

(Confirm the statement – p.783)

Theorem E.5.1: Consider the DARE (above). The following are equivalent: (i) $\{F, H\}$ is detectable and $\{F^s, GQ^{s/2}\}$ is controllable on the unit circle; (ii) the DARE has a stabilizing solution P , i.e., one for which the matrix $F - K_pH$ is stable. Moreover, any such stabilizing solution is unique and positive semi-definite.

6 Callier / Desoer – Linear System Theory

Model – p. 57 – discrete-time system representation

$$x_{k+1} = A_k x_k + B_k u_k$$

$$y_k = C_k x_k + D_k u_k$$

Problem: we have w_k instead of $D_k u_k$. We can always find an appropriate D_k unless $u_k = 0$, which happens with probability 0, presumably, so it should be okay?

Time-invariant: (A, B, C, D) constant in time.

(Thms: pp. 293–294)

Theorem 8.d.62.ii Consider a discrete-time time-invariant system representation $R_d = [A, B, C, D]$. If A and B are real, there exists $F \in \mathbb{R}^{n_i \times n}$ such that

$$\sigma_p(A + BF) \subset \mathcal{D}(0, 1) = \{z \in \mathbb{C} : |z| < 1\}$$

if and only if the pair (A, B) is stabilizable.

Theorem 8.d.65.ii Consider a discrete-time time-invariant system representation $R_d = [A, B, C, D]$. If C and A are real, there exists $L \in \mathbb{R}^{n \times n_o}$ such that

$$\sigma_p(A + LC) \subset \mathcal{D}(0, 1) = \{z \in \mathbb{C} : |z| < 1\}$$

if and only if the pair (C, A) is detectable.

Problem: These theorems are for a time-invariant system. We have time-invariance on everything except D_k due to our w_k issue. This may be a non-issue, but I have no idea.

Translate to (F, G, H, K) notation:

Detectable: $\{F, H\}$ is detectable if and only if $\rho(F + LH) < 1$ for some L .

Stabilizable: $\{F, G\}$ is stabilizable if and only if $\rho(F + GL) < 1$ for some L .

Translate to our problem, $F = I$, $H = A$, $G = I$, $Q = \sigma_{\text{sys}}^2 I$:

Detectable: $\{I, A\}$ is detectable if and only if $\rho(I + LA) < 1$ for some L .

Stabilizable: $\{I, \sigma_{\text{sys}} I\}$ is stabilizable if and only if $\rho(I + \sigma_{\text{sys}} L) < 1$ for some L .

7 Hassibi – PhD Thesis

Results on Riccati equations.

Section 7.3, footnote 1: Def: $\{F, H\}$ is detectable if there exists a matrix K such that $F - KH$ is stable.

Linking this with the definition of detectable from KSHLinear:

$$\exists K \text{ with } F - KH \text{ stable} \equiv \text{detectable} \equiv \{F^*, H^*\} \text{ stabilizable}$$

Lemma 8.7.3 Consider the Riccati recursion with positive semi-definite initial condition

$$P_{i+1} = FP_iF^* + GQG^* - K_{p,i}R_{e,i}K_{p,i}^*, \quad P_0 \succeq 0.$$

If $Q \succ 0$, $R \succ 0$, $\{F, H\}$ is detectable and $\{F - GSR^{-1}H, GQ - GSR^{-1}S^*\}$ is stabilizable then P_i converges to the unique positive semi-definite matrix, P , that satisfies the discrete-time algebraic Riccati equation

$$P = FPF^* + GQG^* - (FPH^* + GS)(R + HPH^*)^{-1}(FPH^* + GS)^*.$$

Theorem 8.7.1 Consider the Riccati recursion

$$P_{i+1} = FP_iF^* + GQG^* - K_{p,i}R_{e,i}K_{p,i}^*, \quad P_0$$

where $R \succ 0$, $Q - SR^{-1}S^* \succ 0$, $\{F, H\}$ is detectable and $\{F - GSR^{-1}H, GQ - GSR^{-1}S^*\}$ is stabilizable. Suppose, moreover, that the initial condition P_0 is such that

$$I + (P^a)^{*/2}P_0(P^a)^{1/2} \succ 0,$$

where P^a is the unique positive semi-definite solution to the dual Riccati recursion. Then P_i converges to the unique positive semi-definite matrix, P , that satisfies the discrete-time algebraic Riccati equation

$$P = FPF^* + GQG^* - (FPH^* + GS)(R + HPH^*)^{-1}(FPH^* + GS)^*.$$

8 Proofs

Lemma 1. Assume that $\{x_t\}$ and $\{y_t\}$ follow the signal model above, $\{t_0, t_0 + 1, t_0 + 2, \dots\}$ is a discrete set of sampling times, only additions to true support ($N_t \subseteq N_{t+1}$ for all t), etc.

Further assume that

- i) The true solution is exactly recovered at the initial time t_0 : $\hat{x}_{t_0} = x_{t_0}$, so $\hat{N}_{t_0} = N_{t_0} = N_0$; *Can we relax this to just the true support is recovered?*
- ii) The maximum support size S_{max} satisfies $S_{max} \leq S_{**} = \max\{s : \delta_{2s}(A) < \sqrt{2} - 1\}$;
- iii) The observation noise w_t is bounded in magnitude: $\|w_t\| < \xi$ for all t and some $\xi > 0$;
- iv) The addition thresholds α_t satisfy $\alpha_t = \alpha = C\xi$ for all t , where

$$C = C(S_{max}) = \frac{4\sqrt{1 + \delta_{2S_{max}}}}{1 - (1 + \sqrt{2})\delta_{2S_{max}}}$$

with $\delta_{2S_{max}} = \delta_{2S_{max}}(A)$; and

- v) The addition delay d satisfies $d > \tau_{det}$, where the detection delay τ_{det} is defined by

$$\tau_{det} = \tau_{det}(\alpha, \varepsilon) = \left\lceil \left(\frac{2\alpha}{\sigma_{sys} \mathcal{Q}^{-1}\left(\frac{(1-\varepsilon)^{1/S_{add}}}{2}\right)} \right)^2 \right\rceil.$$

Here, $\mathcal{Q}^{-1}(x)$ is the inverse of the Gaussian \mathcal{Q} -function, $\mathcal{Q}(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$.

Then

- 1) $\|x_t - \hat{x}_{t, CSres}\|_2 \leq \alpha$ for all sampling times t ;
- 2) there are no false support additions: $\hat{N}_t \subseteq N_t$ for all sampling times t ; and
- 3) $\Pr(E_j | F_j) \geq 1 - \varepsilon$, where $E_j = \{\hat{N}_t = N_t \text{ for all } t \in [t_j + \tau_{det} : t_{j+1} - 1]\}$ and $F_j = \{\hat{N}_{t_j-1} = N_{t_j-1}\}$.

We may want to split claim 3 into its own piece because its proof relies on the other 2 parts, which are proved separately with induction.

Proof. Need to find some way to get Candes Thm 1.3 in here and make the connection that $\hat{x}_{t,\text{CSres}}$ in our notation is x^* in his. Also need to point out that the way we chose α , we have any $C_1\xi \leq C_1(S_{\max})\xi = \alpha$.

To prove claims 1 and 2, we proceed by induction on the value of t .

Consider the base case, where $t = t_0$. Claim 1 follows from Theorem 1.3 in [1] and assumptions (ii), (iii), and (iv) (**Not immediate – need to connect to Candes as above**), and assumption (i) trivially proves claim 2.

Suppose now that claims 1 and 2 are both true for some time $(t - 1)$. We show that the claims are true at time t .

First, we verify claim 1 at time t . Let **TYP0 HERE?** $y_t \rightarrow y_{t,\text{res}}$? **Can probably remove after getting algorithm typed up.**

$$\begin{aligned}\beta_t &= x_t - \hat{x}_{t,\text{init}} \\ \hat{\beta}_t &= \arg \min_{\beta} \|\beta\|_1 \text{ subject to } \|y_{t,\text{res}} - A\beta\|_2 < \xi \\ \hat{x}_{t,\text{CSres}} &= \hat{x}_{t,\text{init}} + \hat{\beta}_t,\end{aligned}$$

where $\hat{x}_{t,\text{init}}$ and $y_{t,\text{res}}$ are defined in the KFCS with LS algorithm and $\hat{x}_{t,\text{init}}$ satisfies $\text{supp}(\hat{x}_{t,\text{init}}) = \hat{N}_{t-1}$.

By the induction hypothesis, $\hat{N}_{t-1} \subseteq N_{t-1}$, and by our model assumptions we have $N_{t-1} \subseteq N_t$. Therefore, $\text{supp}(\beta_t) \subseteq N_t \cup N_{t-1} = N_t$, so $|\text{supp}(\beta_t)| \leq |N_t| \leq S_{\max}$. With this, we can apply Theorem 1.3 in [1] to see that $\|\beta_t - \hat{\beta}_t\|_2 \leq \alpha$ (**AGAIN, need to make this connection**). By the definitions of β_t and $\hat{x}_{t,\text{CSres}}$, we see that $\|\beta_t - \hat{\beta}_t\|_2 = \|x_t - \hat{x}_{t,\text{CSres}}\|_2$, so claim 1 follows.

Next, we verify claim 2 at time t . Suppose that $(x_t)_i = 0$ for some index i , so that $i \notin \text{supp}(x_t) = N_t$. Since $N_{t-1} \subseteq N_t$, we must also have $i \notin N_{t-1}$; by the induction hypothesis, this implies that $i \notin \hat{N}_{t-1}$.

Applying the result of claim 1,

$$|(\hat{x}_{t,\text{CSres}})_i| = |(x_t - \hat{x}_{t,\text{CSres}})_i| \leq \|x_t - \hat{x}_{t,\text{CSres}}\|_2 \leq \alpha.$$

Referring to the algorithm, $\hat{N}_t = \hat{N}_{t-1} \cup \{j : |(\hat{x}_{t,\text{CSres}})_j| > \alpha\}$. Since $i \notin \hat{N}_{t-1}$ and $|(\hat{x}_{t,\text{CSres}})_i| \leq \alpha$, it follows that $i \notin \hat{N}_t$. Thus if $i \notin N_t$, then $i \notin \hat{N}_t$; equivalently, if $i \in \hat{N}_t$, then $i \in N_t$. Therefore, $\hat{N}_t \subseteq N_t$, which proves claim 2 and completes our induction proof.

Now, we prove claim 3. Let $\Delta_t = N_t \setminus \hat{N}_{t-1}$ denote the set of indices of the true support at time t which have not been detected before time t . Suppose that F_j holds, that is, $\hat{N}_{t_j-1} = N_{t_j-1}$.

Since F_j holds, $\Delta_t \subseteq \Delta_{\text{add}, t_j}$ for all $t \in [t_j : t_{j+1} - 1]$.

Let $i \in \Delta_t$ for some $t \in [t_j : t_{j+1} - 1]$ and suppose that $|(x_t)_i| > 2\alpha$. Applying the result from claim 1,

$$0 \leq |(x_t - \hat{x}_{t, \text{CSres}})_i| \leq \|(x_t - \hat{x}_{t, \text{CSres}})\|_2 \leq \alpha < 2\alpha < |(x_t)_i|,$$

so that

$$\begin{aligned} |(\hat{x}_{t, \text{CSres}})_i| &= |(x_t)_i - [(x_t)_i - (\hat{x}_{t, \text{CSres}})_i]| \\ &\geq ||(x_t)_i| - |(x_t - \hat{x}_{t, \text{CSres}})_i|| \\ &= |(x_t)_i| - |(x_t - \hat{x}_{t, \text{CSres}})_i| \\ &> 2\alpha - \alpha \\ &= \alpha. \end{aligned}$$

We see that if $|(x_t)_i| > 2\alpha$, then $|(\hat{x}_{t, \text{CSres}})_i| > \alpha$, so $i \in \hat{N}_t = \hat{N}_{t-1} \cup \{j : |(\hat{x}_{t, \text{CSres}})_j| > \alpha\}$.

If $|(x_t)_i| > 2\alpha$ for all $i \in \Delta_{\text{add}, t_j}$, then $\Delta_t \subseteq \Delta_{\text{add}, t_j} \subseteq \hat{N}_t$; in words, we will detect all “missing” indices at time t , so $\hat{N}_t = N_t$.

From the above discussion, we see that the event $\{|(x_t)_i| > 2\alpha \text{ for all } i \in \Delta_{\text{add}, t_j}\}$ is contained within the event $\{|(x_t)_i| > 2\alpha \text{ for all } i \in \Delta_t \mid F_j\}$, which in turn is contained within the event $\{\hat{N}_t = N_t \mid F_j\}$.

All of the above is still kind of weak in places. It all makes sense in words and is true, but the math / set theory is kind of wonky.

Our model asserts that the entries $(x_t)_i$ of x_t are independent and identically distributed $\mathcal{N}(0, (t - t_j)\sigma_{\text{sys}}^2)$ random variables. With this in mind, we see that

$$\begin{aligned} \Pr(\hat{N}_t = N_t \mid F_j) &\geq \Pr(|(x_t)_i| > 2\alpha \text{ for all } i \in \Delta_t \mid F_j) \\ &\geq \Pr(|(x_t)_i| > 2\alpha \text{ for all } i \in \Delta_{\text{add}, t_j}) \\ &= [\Pr(|(x_t)_1| > 2\alpha)]^{S_{\text{add}}} \\ &= \left[2\mathcal{Q}\left(\frac{2\alpha}{\sigma_{\text{sys}}\sqrt{t - t_j}}\right) \right]^{S_{\text{add}}}. \end{aligned}$$

We examine the particular case where $t = t_j + \tau_{\text{det}}$. In this case,

$$\begin{aligned} \mathbf{Pr}(\hat{N}_{t_j+\tau_{\text{det}}} = N_{t_j+\tau_{\text{det}}} \mid F_j) &\geq \left[2\mathcal{Q} \left(\frac{2\alpha}{\sigma_{\text{sys}}\sqrt{(t_j + \tau_{\text{det}}} - t_j)} \right) \right]^{S_{\text{add}}} \\ &= \left[2\mathcal{Q} \left(\frac{2\alpha}{\sigma_{\text{sys}}\sqrt{\tau_{\text{det}}}} \right) \right]^{S_{\text{add}}} \\ &\geq 1 - \varepsilon, \end{aligned}$$

where the final inequality is easily verified and follows from the ceiling in the definition of τ_{det} and the fact that \mathcal{Q} is a decreasing function.

If $\hat{N}_t = N_t$ for $t = t_j + \tau_{\text{det}}$, then the model assumptions of no support deletions and no support additions until time t_{j+1} , in addition to the result of claim 2, imply that $\hat{N}_t = N_t$ for all $t \in [t_j + \tau_{\text{det}} : t_{j+1} - 1]$, which is exactly the event E_j . Therefore, $\mathbf{Pr}(E_j \mid F_j) = \mathbf{Pr}(\hat{N}_{t_j+\tau_{\text{det}}} = N_{t_j+\tau_{\text{det}}} \mid F_j) \geq 1 - \varepsilon$, which completes the proof. \square

Lemma 2. Assume that $\{x_t\}$ and $\{y_t\}$ follow the signal model above, $\{t_0, t_0 + 1, t_0 + 2, \dots\}$ is a discrete set of sampling times, only additions to true support ($N_t \subseteq N_{t+1}$ for all t), etc.

$$\delta_{S_{\max}}(A) < 1, \alpha_{\text{del}} = 0.$$

Define the event $D = \{\hat{N}_t = N_t = N_* \text{ for all } t \in [t_* : t_{**}]\}$, where N_* is some fixed index set.

At each time t , let $\hat{x}_t = \hat{x}_{t, \text{KFCS}}$ be the KFCS estimate of x_t and let $\tilde{x}_t = \hat{x}_{t, \text{GAKF}}$ be the GAKF estimate of x_t .

Then given any ε and ε_{err} there exists some $\tau_{\text{KF}} = \tau_{\text{KF}}(\varepsilon, \varepsilon_{\text{err}}, N_*)$ such that for all $t \in [t_* + \tau_{\text{KF}} : t_{**}]$, we have $\Pr(\|\tilde{x}_t - \hat{x}_t\|_2^2 \leq \varepsilon_{\text{err}} \mid D) > 1 - \varepsilon$. *[There is an upper-bound condition on ε , detailed in the proof – may want to say “ ε small enough,” or say “given any ε_{err} and any ε small enough.”] Note that if $t_* + \tau_{\text{KF}} > t_{**}$, then this interval is empty and the result is vacuously true.*

Proof. Throughout, we assume that the event D occurs and $t \in [t_* : t_{**}]$, so that all vectors are supported on N_* . **Not true – only x and various others are. y and w are not, could be supported on 1 through n . Point this out.**

We consider all variables and parameters only along the support set N_* , but for simplicity of notation we will leave off the subscript N_* where possible. Thus, $\nu_t = (\nu_t)_{N_*}$, $A = A_{[1:n], N_*}$, $\hat{x}_t = (\hat{x}_t)_{N_*}$, $K_t = (K_t)_{N_*, [1:n]}$, $P_{t|t-1} = (P_{t|t-1})_{N_*, N_*}$, $J_t = (J_t)_{N_*, N_*}$, and analogously for \tilde{x}_t , \tilde{K}_t , $\tilde{P}_{t|t-1}$, and \tilde{J}_t .

For $t > t_*$, both KFCS and GAKF run the same fixed-dimensional and fixed-parameter Kalman filter for $(x_t)_{N_*}$, but with different initial conditions. **Elaborate + possibly combine with first paragraph about DARE stuff below**

↓ ————— **moved to enhance the flow of the proof**

Notice that for $t \in [t_* : t_{**}]$ our model is a discrete-time time-invariant linear system with $(F, G, H, K) = (I, I, A, I)$. We can apply elements of linear system theory **discussed in SECTION** to generate results relating to $P_{t|t-1}$. **As noted above: technically K is not time-invariant because of w_t not being Ku_t , so this may break down.**

We see that

$$\begin{aligned} P_{t+1|t} &= P_t + Q \\ &= (I - K_t A) P_{t|t-1} + Q \\ &= P_{t|t-1} + Q - P_{t|t-1} A' (A P_{t|t-1} A' + R)^{-1} A P_{t|t-1}, \end{aligned}$$

which is a discrete algebraic Riccati recursion (DARR) with $F = I$, $G = I$, $Q = \sigma_{\text{sys}}^2 I_{N_*}$, $R = \sigma_{\text{obs}}^2 I$, and $S = 0$. **Verify Q, R.** Note that Q and R are fixed in time since we assume that D occurs.

Since $|N_*| \leq S_{\max}$ and $\delta_{S_{\max}} < 1$, $A = (A_{[1:n], N_*})$ is full rank. **Need to link this to detectability.**

Since $\sigma_{\text{sys}} > 0$, $Q^{1/2} = \sigma_{\text{sys}} I$ is full rank, where $Q^{1/2}$ denotes the matrix square root of Q . We see that the matrix $L = -\frac{1}{\sigma_{\text{sys}}+1}I$ satisfies $\rho(I + \sigma_{\text{sys}}IL) = \rho\left[\left(\frac{1}{\sigma_{\text{sys}}+1}\right)I\right] < 1$, so $\{I, \sigma_{\text{sys}}I\}$ is stabilizable. **Previously had thought we need $\sigma_{\text{sys}} \neq 1$ (see the rank definition - rank of a 0 matrix is never n), but I think it cleaned itself up...**

Therefore, by **HASSIBI PHD RESULT**, the DARR converges to a positive semi-definite matrix P_* . This implies that $K_t \rightarrow K_* = P_*A'(AP_*A' + \mathbf{R})^{-1}$ and $J_t \rightarrow J_* = (I - K_*A)$. Further, by **LINEAR THM E.5.1**, $\rho(J_*) < 1$.

Since GAKF and KFCS run the same Kalman filter, these results also apply to the GAKF iterates, i.e. $\tilde{P}_{t|t-1} \rightarrow P_*$, $\tilde{K}_t \rightarrow K_*$, and $\tilde{J}_t \rightarrow J_*$.

Let $\rho = \rho(J_*)$ and let $\varepsilon_0 = (1 - \rho)/2$. A standard result from linear algebra states that there exists a matrix norm $\|\cdot\|_\rho$ such that $\|J_*\|_\rho \leq \rho + \varepsilon_0 = (1 + \rho)/2 < 1$. Further, by the equivalence of matrix norms on a finite-dimensional space, there exists some constant $c_{\rho,2}$ such that $\|M\|_\rho \leq c_{\rho,2}\|M\|_2$ for any matrix M .

Let $0 < \varepsilon < (1 - \rho)/4$ be arbitrary. **— Here is our upper bound on ε . — UPDATE: This isn't ε anymore. Figure out how the bound affects things.—**

The convergence of K_t and \tilde{K}_t to the common limit of K_* implies that there exists some τ_ε^K such that for all $t \geq \tau_\varepsilon^K$, $\|K_t - \tilde{K}_t\|_2 < \varepsilon$. Similarly, there exists some τ_ε^J for which $\|J_t - \tilde{J}_t\|_2 < \varepsilon$ for all $t \geq \tau_\varepsilon^J$, and there exists some τ_ε^R such that $\|J_t\|_\rho < \|J_*\|_\rho + \varepsilon$ for all $t \geq \tau_\varepsilon^R$. Let $t_\varepsilon \geq \max\{\tau_\varepsilon^K, \tau_\varepsilon^J, \tau_\varepsilon^R\}$; then for all $t \geq t_\varepsilon$, we have $\|K_t - \tilde{K}_t\|_2 < \varepsilon$, $\|J_t - \tilde{J}_t\|_2 < \varepsilon$, and $\|J_t\|_\rho < \|J_*\|_\rho + \varepsilon = (1 + \rho)/2 + \varepsilon < (1 + \rho)/2 + (1 - \rho)/4 = (3 + \rho)/4 < 1$. **This is verbose, and probably unnecessarily so; this result is obvious. Can probably stick with something closer to the original, below.**

The convergence results above and standard analysis techniques can be used to show that there exists some $t_\varepsilon > t_*$ such that for all $t \geq t_\varepsilon$, we have $\|K_t - \tilde{K}_t\|_2 < \varepsilon$, $\|J_t - \tilde{J}_t\|_2 < \varepsilon$, and $\|J_t\|_\rho \leq \|J_*\|_\rho + \varepsilon < (1 + \rho)/2 + (1 - \rho)/4 = (3 + \rho)/4 < 1$.

Point out that t_ε is independent of $y...$

Problem: NV proof says \hat{x}_{t_*} is independent of y_{t_*} , but by definition it's not. AB draft: $t_* - 1$. So I agree that we're independent of $y_1 \dots y_{t_*-1}$, but we are dependent on $y_{t_*} \dots y_t$ because $\hat{x}_t = J_t \hat{x}_{t-1} + K_t y_t$ for $t > t_*$. **All of this independence stuff needs to be very carefully worked and verified.**

$P_{t|t-1}$ bounded...?

↑ ————— /moved

Let $\hat{e}_t = x_t - \hat{x}_t$ and $\tilde{e}_t = x_t - \tilde{x}_t$. Define $\text{diff}_t = \hat{e}_t - \tilde{e}_t$ and notice that $\text{diff}_t = \tilde{x}_t - \hat{x}_t$.

Let $t > t_\varepsilon > t_*$. By the KFCS with LS algorithm and the model, we see that

$$\begin{aligned}
\hat{e}_t &= x_t - \hat{x}_t \\
&= (x_{t-1} + \nu_t) - (J_t \hat{x}_{t-1} + K_t y_t) \\
&= x_{t-1} + \nu_t - J_t \hat{x}_{t-1} - K_t (Ax_t + w_t) \\
&= x_{t-1} + \nu_t - J_t \hat{x}_{t-1} - K_t A(x_{t-1} + \nu_t) - K_t w_t \\
&= (I - K_t A)x_{t-1} - J_t \hat{x}_{t-1} + (I - K_t A)\nu_t - K_t w_t \\
&= J_t(x_{t-1} - \hat{x}_{t-1}) + J_t \nu_t - K_t w_t \\
&= J_t \hat{e}_{t-1} + J_t \nu_t - K_t w_t.
\end{aligned}$$

Similarly, using the GAKF algorithm and the model, we can verify that

$$\tilde{e}_t = \tilde{J}_t \tilde{e}_{t-1} + \tilde{J}_t \nu_t - \tilde{K}_t w_t.$$

Combining these results yields

$$\text{diff}_t = J_t \text{diff}_{t-1} + (J_t - \tilde{J}_t)(\tilde{e}_{t-1} + \nu_t) + (\tilde{K}_t - K_t)w_t.$$

Let $u_t = (J_t - \tilde{J}_t)(\tilde{e}_{t-1} + \nu_t) + (\tilde{K}_t - K_t)w_t$, so that $\text{diff}_t = J_t \text{diff}_{t-1} + u_t$. Recursively applying this identity, we see that

$$\begin{aligned}
\text{diff}_t &= J_t \text{diff}_{t-1} + u_t \\
&= J_t (J_{t-1} \text{diff}_{t-2} + u_{t-1}) + u_t \\
&= J_t J_{t-1} \text{diff}_{t-2} + J_t u_{t-1} + u_t \\
&= J_t J_{t-1} (J_{t-2} \text{diff}_{t-3} + u_{t-2}) + J_t u_{t-1} + u_t \\
&= J_t J_{t-1} J_{t-2} \text{diff}_{t-3} + J_t J_{t-1} u_{t-2} + J_t u_{t-1} + u_t \\
&\vdots \\
&= J_t J_{t-1} \cdots J_{t_\varepsilon+1} \text{diff}_{t_\varepsilon} + J_t J_{t-1} \cdots J_{t_\varepsilon+2} u_{t_\varepsilon+1} + \dots + J_t u_{t-1} + u_t.
\end{aligned}$$

If we define $M_k^t = J_t J_{t-1} \cdots J_{k+1} J_k$ for $k \leq t$, then we can more compactly write

$$\text{diff}_t = M_{t_\varepsilon+1}^t \text{diff}_{t_\varepsilon} + u_t + \sum_{k=t_\varepsilon+1}^{t-1} M_{k+1}^t u_k.$$

Therefore, applying the triangle and Cauchy-Schwarz inequalities for expectation and

noting that the matrices $\{M_k^t\}$ are deterministic,

$$\begin{aligned}
\mathbb{E} [\|\text{diff}_t\|_2^2 \mid D]^{\frac{1}{2}} &= \mathbb{E} \left[\left\| M_{t_\varepsilon+1}^t \text{diff}_{t_\varepsilon} + u_t + \sum_{k=t_\varepsilon+1}^{t-1} M_{k+1}^t u_k \right\|_2^2 \mid D \right]^{\frac{1}{2}} \\
&\leq \mathbb{E} [\|M_{t_\varepsilon+1}^t \text{diff}_{t_\varepsilon}\|_2^2 \mid D]^{\frac{1}{2}} + \mathbb{E} [\|u_t\|_2^2 \mid D]^{\frac{1}{2}} + \sum_{k=t_\varepsilon+1}^{t-1} \mathbb{E} [\|M_{k+1}^t u_k\|_2^2 \mid D]^{\frac{1}{2}} \\
&\leq \mathbb{E} [\|M_{t_\varepsilon+1}^t\|_2^2 \mid D]^{\frac{1}{2}} \mathbb{E} [\|\text{diff}_{t_\varepsilon}\|_2^2 \mid D]^{\frac{1}{2}} + \\
&\quad \mathbb{E} [\|u_t\|_2^2 \mid D]^{\frac{1}{2}} + \sum_{k=t_\varepsilon+1}^{t-1} \mathbb{E} [\|M_{k+1}^t\|_2^2 \mid D]^{\frac{1}{2}} \mathbb{E} [\|u_k\|_2^2 \mid D]^{\frac{1}{2}} \\
&\leq \|M_{t_\varepsilon+1}^t\|_2 \mathbb{E} [\|\text{diff}_{t_\varepsilon}\|_2^2 \mid D]^{\frac{1}{2}} + \\
&\quad \left(1 + \sum_{k=t_\varepsilon+1}^{t-1} \|M_{k+1}^t\|_2 \right) \max_{\tau \in [t_\varepsilon+1:t]} \left\{ \mathbb{E} [\|u_\tau\|_2^2 \mid D]^{\frac{1}{2}} \right\}.
\end{aligned}$$

Let $a = (3 + \rho)/4$ and recall that $\|J_k\|_\rho \leq a < 1$ for any $k \geq t_\varepsilon$. This implies that

$$\begin{aligned}
\|M_k^t\|_\rho &= \|J_t J_{t-1} \cdots J_k\|_\rho \\
&\leq \|J_t\|_\rho \|J_{t-1}\|_\rho \cdots \|J_k\|_\rho \\
&\leq a^{t-k+1},
\end{aligned}$$

so $\|M_k^t\|_2 \leq c_{\rho,2} a^{t-k+1}$. With this, we see that

$$\begin{aligned}
\left(1 + \sum_{k=t_\varepsilon+1}^{t-1} \|M_{k+1}^t\|_2 \right) &\leq \left(1 + \sum_{k=t_\varepsilon+1}^{t-1} c_{\rho,2} a^{t-k} \right) \\
&\leq \max\{1, c_{\rho,2}\} \cdot \sum_{\ell=0}^{\infty} a^\ell \\
&= \max\{1, c_{\rho,2}\} \cdot \frac{1}{1-a}.
\end{aligned}$$

Let $\tau \in [t_\varepsilon + 1 : t]$ be arbitrary and consider

$$\begin{aligned}
\mathbb{E} [\|u_\tau\|_2^2 \mid D]^{\frac{1}{2}} &= \mathbb{E} \left[\left\| (J_\tau - \tilde{J}_\tau)(\tilde{e}_{\tau-1} + \nu_\tau) + (\tilde{K}_\tau - K_\tau)w_\tau \right\|_2^2 \mid D \right]^{\frac{1}{2}} \\
&\leq \|J_\tau - \tilde{J}_\tau\|_2 \mathbb{E} [\|\tilde{e}_{\tau-1} + \nu_\tau\|_2^2 \mid D]^{\frac{1}{2}} + \|\tilde{K}_\tau - K_\tau\|_2 \mathbb{E} [\|w_\tau\|_2^2 \mid D]^{\frac{1}{2}} \\
&< \varepsilon \cdot \mathbb{E} [\|\tilde{e}_{\tau-1} + \nu_\tau\|_2^2 \mid D]^{\frac{1}{2}} + \varepsilon \cdot \mathbb{E} [\|w_\tau\|_2^2 \mid D]^{\frac{1}{2}} \\
&\leq \varepsilon \left(\mathbb{E} [\|\tilde{e}_{\tau-1}\|_2^2 \mid D]^{\frac{1}{2}} + \mathbb{E} [\|\nu_\tau\|_2^2 \mid D]^{\frac{1}{2}} + \mathbb{E} [\|w_\tau\|_2^2 \mid D]^{\frac{1}{2}} \right),
\end{aligned}$$

where we have used the triangle and Cauchy-Schwarz inequalities for expectation and the fact that $\tau > t_\varepsilon$. **More detail to emphasize this is the bound on J and K?**

Recall that $\nu_\tau = (\nu_\tau)_{N_*}$ and $\text{cov}(\nu_\tau, \nu_\tau) = \mathbb{E}[\nu_\tau \nu_\tau']$ is independent of D . **Elaborate why?** Notice that

$$\begin{aligned} \mathbb{E} [\|\nu_\tau\|_2^2 \mid D] &= \text{tr} (\mathbb{E} [\nu_\tau' \nu_\tau \mid D]) \\ &= \mathbb{E} [\text{tr}(\nu_\tau' \nu_\tau) \mid D] \\ &= \mathbb{E} [\text{tr}(\nu_\tau \nu_\tau') \mid D] \\ &= \text{tr} (\mathbb{E} [\nu_\tau \nu_\tau' \mid D]) \\ &= \text{tr} (\mathbb{E} [\nu_\tau \nu_\tau']) \\ &= \text{tr} (\sigma_{\text{sys}}^2 I_{|N_*| \times |N_*|}) \\ &= |N_*| \sigma_{\text{sys}}^2. \end{aligned}$$

A similar computation proves that $\mathbb{E} [\|w_\tau\|_2^2 \mid D] = n\sigma_{\text{obs}}^2$. Recalling that $\mathbb{E} [\|\tilde{e}_\tau\|_2^2 \mid D] < B$ **verify < or \leq** , we have

$$\mathbb{E} [\|u_\tau\|_2^2 \mid D]^{\frac{1}{2}} < \varepsilon \left(\sqrt{B} + \sqrt{|N_*| \sigma_{\text{sys}}^2} + \sqrt{n\sigma_{\text{obs}}^2} \right).$$

Since $\tau \in [t_\varepsilon + 1 : t]$ was arbitrary, we conclude that

$$\max_{\tau \in [t_\varepsilon + 1 : t]} \left\{ \mathbb{E} [\|u_\tau\|_2^2 \mid D]^{\frac{1}{2}} \right\} < \varepsilon \left(\sqrt{B} + \sqrt{|N_*| \sigma_{\text{sys}}^2} + \sqrt{n\sigma_{\text{obs}}^2} \right).$$

Combining these results with **reference prev equations....**, we see that

$$\mathbb{E} [\|\text{diff}_t\|_2^2 \mid D]^{\frac{1}{2}} < c_{\rho,2} a^{t-t_\varepsilon} \mathbb{E} [\|\text{diff}_{t_\varepsilon}\|_2^2 \mid D]^{\frac{1}{2}} + C\varepsilon,$$

where $C = \max\{1, c_{\rho,2}\} \cdot \frac{1}{1-a} \cdot \left(\sqrt{B} + \sqrt{|N_*| \sigma_{\text{sys}}^2} + \sqrt{n\sigma_{\text{obs}}^2} \right)$.

Problem: why is $\mathbb{E} [\|\text{diff}_{t_\varepsilon}\|_2^2 \mid D]^{\frac{1}{2}}$ bounded? NV ignores it, and AB says "because they're finite," which needs more explanation. Moving on, assume it's finite / bounded.

If

$$t_{\text{ms}} = t_\varepsilon + \log_a \left(\frac{C\varepsilon}{c_{\rho,2} \mathbb{E} [\|\text{diff}_{t_\varepsilon}\|_2^2 \mid D]^{\frac{1}{2}}} \right),$$

then we see that for all $t \geq t_{\text{ms}}$,

$$\mathbb{E} [\|\tilde{x}_t - \hat{x}_t\|_2^2 \mid D]^{\frac{1}{2}} = \mathbb{E} [\|\text{diff}_t\|_2^2 \mid D]^{\frac{1}{2}} < 2C\varepsilon,$$

and since ε can be chosen arbitrarily small we conclude that \hat{x}_t converges to \tilde{x}_t in mean square.

FIX THE REST OF THIS PROOF! We'll need to go through and change all the ε to some other symbol, then change X to ε , then add more detail.

Now, by Markov's inequality, for all $t \geq t_{\text{ms}}$,

$$\Pr \left(\|\tilde{x}_t - \hat{x}_t\|_2^2 > \varepsilon_{\text{err}} \mid D \right) \leq \frac{\mathbb{E} [\|\tilde{x}_t - \hat{x}_t\|_2^2 \mid D]}{\varepsilon_{\text{err}}} < \frac{4C^2\varepsilon^2}{\varepsilon_{\text{err}}}.$$

Given any $X > 0$, choose $\varepsilon = \frac{\sqrt{X \cdot \varepsilon_{\text{err}}}}{2C}$. Then

$$\Pr \left(\|\tilde{x}_t - \hat{x}_t\|_2^2 \leq \varepsilon_{\text{err}} \mid D \right) < 1 - X$$

for all $t \geq t_{\text{ms}} = t_* + \tau_{\text{KF}}$.

□

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