KFCS Theory

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1 Model

Major change: time indexing redone to match NV original. t_0 is now the first addition and we assume there's an initial $(t_0 - 1)$ step.

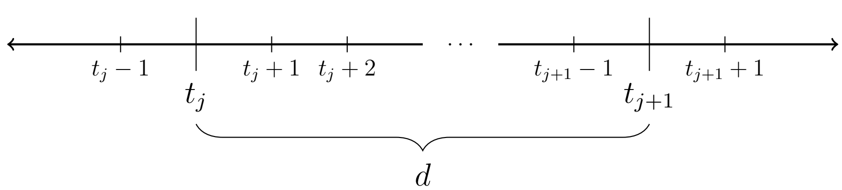
At each time $t \geq t_0$, we have

$$\begin{aligned} x_t &= x_{t-1} + \nu_t \\ y_t &= Ax_t + w_t \end{aligned}$$

Here, $\mathbb{E}[w_t] = \mathbf{0}$, $\text{cov}(w_t) = \mathbb{E}[w_t w_t'] = R = \sigma^2_{\text{obs}} I_{n\times n}$, iid and independent of x_t; $x_{t_0-1} \sim \mathcal{N}(\mathbf{0}, \sigma^2_{\text{sys},0}\mathcal{I}_{N_{t_0-1}})$; and $\nu_t \sim \mathcal{N}(\mathbf{0}, \sigma^2_{\text{sys}}\mathcal{I}_{N_t})$ iid. for $t \geq t_0$.

$y_t, w_t \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times m}$, $x_t, \nu_t \in \mathbb{R}^m$.

Time indices are discrete. Make the distinction between sampling times (used) and continuous time (not used).



For $j \geq 0$, we have the addition times $\{t_j\}$. The initial time is $t = (t_0 - 1)$. At the addition times $t_j = t_0 + jd$, the support of x_t changes: $N_t = N_{t_j}$ for all $t \in [t_j : t_{j+1} - 1]$, and $N_{t_j} \subset N_{t_{j+1}}$.

2 Algorithm – KFCS

This algorithm applies to the case where there are no support deletions.

Input: σ_{sys}, σ_{obs}, $\sigma_{\text{sys},0}$, A_{t_0-1}, A, $\{t_j\}$, $\{N_t\}$, $\{y_t\}$, ξ

$\hat{x}_{t_0-1} = \mathbf{0}$
$\hat{N}_{t_0-1} = \emptyset$
$P_{t_0-1} = 0$

for $t \geq t_0$ **do**

$Q_t = \sigma_{\text{sys}}^2 \mathcal{I}_{\hat{N}_{t-1}}$
$P_{t|t-1} = P_{t-1} + Q_t$
$K_t = P_{t|t-1} A' \left(A P_{t|t-1} A' + \sigma_{\text{obs}}^2 I \right)^{-1}$
$J_t = I - K_t A$
$P_t = J_t P_{t|t-1}$
$\hat{x}_{t,\text{init}} = J_t \hat{x}_{t-1} + K_t y_t$

$y_{t,\text{res}} = y_t - A \hat{x}_{t,\text{init}}$
$\hat{\beta}_t = \arg\min_\beta \|\beta\|_1$ subject to $\|y_{t,\text{res}} - A\beta\|_2 < \xi$
$\hat{x}_{t,\text{CSres}} = \hat{x}_{t,\text{init}} + \hat{\beta}_t$

$\Delta_A = \{k : |(\hat{x}_{t,\text{CSres}})_k| > \alpha\}$
$\hat{N}_t = \hat{N}_{t-1} \cup \Delta_A$

if $\Delta_A = \emptyset$ **then**

$\hat{x}_t = \hat{x}_{t,\text{init}}$

else

$\hat{x}_t = \mathbf{0}$
$(\hat{x}_t)_{\hat{N}_t} = (A_{[1:n],\hat{N}_t})^\dagger y_t$
$P_t = 0_{m \times m}$
$(P_t)_{\hat{N}_t,\hat{N}_t} = \left[(A_{[1:n],\hat{N}_t})' (A_{[1:n],\hat{N}_t}) \right]^{-1} \sigma_{\text{obs}}^2 I_{|\hat{N}_t| \times |\hat{N}_t|}$

end

end

Algorithm 1: Kalman-Filtered Compressed Sensing (KFCS)

3 Algorithm – Genie-Aided Kalman Filtering (GAKF)

This algorithm applies to the case where there are no support deletions.

Input: σ_{sys}, σ_{obs}, $\sigma_{\text{sys},0}$, A, $\{t_j\}$, $\{N_t\}$, $\{y_t\}$

$\tilde{x}_{t_0-1} = \mathbf{0}$
$\tilde{P}_{t_0-1} = 0$

for $t \geq t_0$ **do**

$\tilde{Q}_t = \sigma_{\text{sys}}^2 \mathcal{I}_{N_t}$

$\tilde{P}_{t|t-1} = \tilde{P}_{t-1} + \tilde{Q}_t$
$\tilde{K}_t = \tilde{P}_{t|t-1} A' \left(A \tilde{P}_{t|t-1} A' + \sigma_{\text{obs}}^2 I \right)^{-1}$
$\tilde{J}_t = I - \tilde{K}_t A$
$\tilde{P}_t = \tilde{J}_t \tilde{P}_{t|t-1}$
$\tilde{x}_t = \tilde{J}_t \tilde{x}_{t-1} + \tilde{K}_t y_t$

end

Algorithm 2: Genie-Aided Kalman Filter (GAKF)

4 Candes RIP – C_1 Computation for α

[1], Theorem 1.3: Suppose $y = Ax + \eta$, $|\operatorname{supp}(x)| = s$, $\delta_{2s} = \delta_{2s}(A) < \sqrt{2} - 1$, and $\|\eta\|_2 \leq \xi$. Then

$$\hat{x} = \arg\min_z \|z\|_1 \text{ subject to } \|y - Az\|_2 \leq \xi$$

satisfies

$$\|x - \hat{x}\|_2 \leq C_1(s)\xi,$$

where

$$C_1(s) = \frac{4\sqrt{1 + \delta_{2s}}}{1 - (1 + \sqrt{2})\delta_{2s}} \ .$$

Claim / Note: It can be shown that C_1 is an increasing function of δ_{2s}, and δ_{2s} is an increasing function of s, so C_1 is an increasing function of s.

For any support size s in this paper, we will have $s \leq S_{\max}$ and thus $C_1(s) \leq C_1(S_{\max})$.

5 Linear Systems Theory

5.1 Definitions

We present some basic definitions from linear systems theory. These can be found in [3], Appendix C. Throughout, let $F, G, H \in \mathbb{R}^{n\times n}$.

A matrix F is **stable** if $\rho(F) < 1$.

The pair $\{F, G\}$ is **controllable** if the matrix $[G, FG, \dots, F^{n-1}G]$ is full rank n. An equivalent characterization of controllability is that $\text{rank}([\lambda I - F,\ G]) = n$ for all eigenvalues λ of F.

The pair $\{F, G\}$ is **stabilizable** if $\text{rank}([\lambda I - F,\ G]) = n$ for all eigenvalues λ of F with $|\lambda| \geq 1$.

The pair $\{F, H\}$ is **detectable** if and only if $\{F', H'\}$ is stabilizable.

Consider the case where $F = I$. Then $\lambda = 1$ is the only eigenvalue of $F = F'$ and the matrix $[\lambda I - F,\ G] = [0, G]$ has rank n if and only if G has rank n. Therefore, if G is full rank, then $\{I, G\}$ is controllable and stabilizable. Additionally, since $\text{rank}(H) = \text{rank}(H')$, we can use the same argument to conclude that $\{I, H\}$ is detectable if H is full rank.

5.2 Theoretical Results

Here we present two important theoretical results from linear systems theory.

The general form of a **discrete-time algebraic Riccati equation (DARE)** is

$$P = FPF' + GQG' - (FPH' + GS)(R + HPH')^{-1}(FPH' + GS)', \tag{1}$$

where $P, F, G, H, Q, R, S \in \mathbb{R}^{n\times n}$.

[2], Theorem 7.5.1.b: Consider the DARE (1), where $\{F, H\}$ is detectable and

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \succ 0.$$

If, in addition, $\{F - GSR^{-1}H, GQ - GSR^{-1}S'\}$ is stabilizable, then the DARE always has a unique Hermitian and positive semi-definite stabilizing solution P such that $F - K_pH$ is stable, where $K_p = (FPH' + GS)(R + HPH')^{-1}$.

The general form of a **discrete-time algebraic Riccati recursion (DARR)** is

$$P_{i+1} = FP_iF' + GQG' - K_{p,i}R_{e,i}K'_{p,i}, \quad i \geq 0 \tag{2}$$

where $K_{p,i} = (FP_iH' + GS)(R + HP_iH')^{-1}$, $R_{e,i} = R + HP_iH'$, and $\{P_k\}, F, G, H, Q, R, S \in \mathbb{R}^{n\times n}$.

[2], Lemma 8.7.3: Consider the Riccati recursion (2) with positive semi-definite initial condition $P_0 \succeq 0$. If $Q \succ 0$, $R \succ 0$, $\{F, H\}$ is detectable and $\{F - GSR^{-1}H,\, GQ - GSR^{-1}S'\}$ is stabilizable then P_i converges to the unique positive semi-definite matrix, P, that satisfies the discrete-time algebraic Riccati equation (1).

6 Proofs

Lemma 1. *Assume that $\{x_t\}$ and $\{y_t\}$ follow the signal model above, $\{t\}$ is a discrete set of sampling times, only additions to true support ($N_t \subseteq N_{t+1}$ for all t), etc.*

Further assume that

i) The true solution is exactly recovered at the initial time $t = (t_0 - 1)$: $\hat{x}_{t_0-1} = x_{t_0-1}$, so $\hat{N}_{t_0-1} = N_{t_0-1}$; ***Can we relax this to just the true support is recovered?***

ii) The maximum support size S_{max} satisfies $S_{max} \leq S_{**} = \max\{s : \delta_{2s}(A) < \sqrt{2} - 1\}$;

iii) The observation noise w_t is bounded in magnitude: $\|w_t\|_2 < \xi$ for all t and some $\xi > 0$;

iv) The addition thresholds α_t satisfy $\alpha_t = \alpha = C\xi$ for all t, where

$$C = C(S_{max}) = \frac{4\sqrt{1 + \delta_{2S_{max}}}}{1 - \left(1 + \sqrt{2}\right) \delta_{2S_{max}}}$$

with $\delta_{2S_{max}} = \delta_{2S_{max}}(A)$; and

v) The addition delay d satisfies $d > \tau_{det}$, where the detection delay τ_{det} is defined by

$$\tau_{det} = \tau_{det}(\alpha, \varepsilon) = \left\lceil \left(\frac{2\alpha}{\sigma_{sys} \mathcal{Q}^{-1}\left(\frac{(1-\varepsilon)^{1/S_{add}}}{2}\right)} \right)^2 - 1 \right\rceil.$$

Here, $\mathcal{Q}^{-1}(x)$ is the inverse of the Gaussian $\mathcal{Q}$-function, $\mathcal{Q}(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt$.

Then

1) $\|x_t - \hat{x}_{t,CSres}\|_2 \leq \alpha$ for all sampling times $t \geq (t_0 - 1)$;

2) There are no false support additions: $\hat{N}_t \subseteq N_t$ for all $t \geq (t_0 - 1)$; and

3) For any $j \geq 0$, $\boldsymbol{Pr}(\mathbf{E_j} \,|\, \mathbf{F_j}) \geq 1 - \varepsilon$, where $\mathbf{E_j} = \{\hat{N}_t = N_t \text{ for all } t \in [t_j + \tau_{det} : t_{j+1} - 1]\}$, $\mathbf{F_j} = \{\hat{N}_{t_j-1} = N_{t_j-1}\}$, and $\varepsilon > 0$ is arbitrary.

Proof. **Need to find some way to get Candes Thm 1.3 in here and make the connection that $\hat{x}_{t,\mathbf{CSres}}$ in our notation is $x^\star$ in his. Also need to point out that the way we chose α, we have any $C_1\xi \leq C_1(S_{\mathbf{max}})\xi = \alpha$.**

To prove claims 1 and 2, we proceed by induction on the value of t.

Consider the base case, where $t = (t_0 - 1)$. Claim 1 follows from [1], Theorem 1.3 and assumptions (ii), (iii), and (iv) **(Not immediate – need to connect to Candes as above)**, and assumption (i) trivially proves claim 2.

Suppose now that claims 1 and 2 are both true for some time $(t-1)$. We show that the claims are true at time t.

First, we verify claim 1 at time t. Referring to Algorithm 1, we have

$$\begin{aligned}
\beta_t &= x_t - \hat{x}_{t,\text{init}} \\
\hat{\beta}_t &= \arg\min_{\beta} \|\beta\|_1 \text{ subject to } \|y_{t,\text{res}} - A\beta\|_2 < \xi \\
\hat{x}_{t,\text{CSres}} &= \hat{x}_{t,\text{init}} + \hat{\beta}_t,
\end{aligned}$$

where $\text{supp}(\hat{x}_{t,\text{init}}) = \hat{N}_{t-1}$.

By the induction hypothesis, $\hat{N}_{t-1} \subseteq N_{t-1}$, and by our model assumptions we have $N_{t-1} \subseteq N_t$. Therefore, $\text{supp}(\beta_t) \subseteq N_t \cup N_{t-1} = N_t$, so $|\text{supp}(\beta_t)| \leq |N_t| \leq S_{\max}$. With this, we can apply [1], Theorem 1.3 to see that $\|\beta_t - \hat{\beta}_t\|_2 \leq \alpha$ **(AGAIN, need to make this connection)**. By the definitions of β_t and $\hat{x}_{t,\text{CSres}}$, we see that $\|\beta_t - \hat{\beta}_t\|_2 = \|x_t - \hat{x}_{t,\text{CSres}}\|_2$, so claim 1 follows.

Next, we verify claim 2 at time t. Suppose that $(x_t)_i = 0$ for some index i, so that $i \notin \text{supp}(x_t) = N_t$. Since $N_{t-1} \subseteq N_t$, we must also have $i \notin N_{t-1}$; by the induction hypothesis, this implies that $i \notin \hat{N}_{t-1}$.

Applying the result of claim 1,

$$|(\hat{x}_{t,\text{CSres}})_i| = |(x_t - \hat{x}_{t,\text{CSres}})_i| \leq \|x_t - \hat{x}_{t,\text{CSres}}\|_2 \leq \alpha.$$

Referring to Algorithm 1, $\hat{N}_t = \hat{N}_{t-1} \cup \{k : |(\hat{x}_{t,\text{CSres}})_k| > \alpha\}$. Since $i \notin \hat{N}_{t-1}$ and $|(\hat{x}_{t,\text{CSres}})_i| \leq \alpha$, it follows that $i \notin \hat{N}_t$. Thus if $i \notin N_t$, then $i \notin \hat{N}_t$; equivalently, if $i \in \hat{N}_t$, then $i \in N_t$. Therefore, $\hat{N}_t \subseteq N_t$, which proves claim 2 and completes our induction proof.

Now, we prove claim 3. Let $\Delta_t = N_t \setminus \hat{N}_{t-1}$ denote the set of indices of the true support at time t which have not been detected before time t. Fix $j \geq 0$ and suppose that $\mathbf{F_j}$ holds, that is, $\hat{N}_{t_j-1} = N_{t_j-1}$.

Since $\mathbf{F_j}$ holds, $\Delta_t \subseteq \Delta_{\text{add},t_j}$ for all $t \in [t_j : t_{j+1} - 1]$.

Let $i \in \Delta_t$ for some $t \in [t_j : t_{j+1} - 1]$ and suppose that $|(x_t)_i| > 2\alpha$. Applying the result from claim 1,

$$0 \leq |(x_t - \hat{x}_{t,\text{CSres}})_i| \leq \|x_t - \hat{x}_{t,\text{CSres}}\|_2 \leq \alpha < 2\alpha < |(x_t)_i|,$$

so that

$$\begin{aligned}
|(\hat{x}_{t,\text{CSres}})_i| &= \left|(x_t)_i - [(x_t)_i - (\hat{x}_{t,\text{CSres}})_i]\,\right| \\
&\geq \left||(x_t)_i| - |(x_t - \hat{x}_{t,\text{CSres}})_i|\right| \\
&= |(x_t)_i| - |(x_t - \hat{x}_{t,\text{CSres}})_i| \\
&> 2\alpha - \alpha \\
&= \alpha.
\end{aligned}$$

We see that if $|(x_t)_i| > 2\alpha$, then $|(\hat{x}_{t,\text{CSres}})_i| > \alpha$, so $i \in \hat{N}_t = \hat{N}_{t-1} \cup \{k : |(\hat{x}_{t,\text{CSres}})_k| > \alpha\}$.

If $|(x_t)_i| > 2\alpha$ for all $i \in \Delta_{\text{add},t_j}$, then $\Delta_t \subseteq \Delta_{\text{add},t_j} \subseteq \hat{N}_t$; in words, we will detect all "missing" indices at time t, so $\hat{N}_t = N_t$.

From the above discussion, we see that the event $\{|(x_t)_i| > 2\alpha \text{ for all } i \in \Delta_{\text{add},t_j}\}$ is contained within the event $\{|(x_t)_i| > 2\alpha \text{ for all } i \in \Delta_t \mid \mathbf{F_j}\}$, which in turn is contained within the event $\{\hat{N}_t = N_t \mid \mathbf{F_j}\}$.

All of the above is still kind of weak in places. It all makes sense in words and is true, but the math / set theory is kind of wonky.

Our model asserts that the entries $(x_t)_i$ for $i \in \Delta_{\text{add},t_j}$ are independent and identically distributed $\mathcal{N}\left(0, (t - t_j + 1)\sigma_{\text{sys}}^2\right)$ random variables. With this in mind, we see that

$$\begin{aligned}
\mathbf{Pr}\left(\hat{N}_t = N_t \,\middle|\, \mathbf{F_j}\right) &\geq \mathbf{Pr}\left(|(x_t)_i| > 2\alpha \text{ for all } i \in \Delta_t \,|\, \mathbf{F_j}\right) \\
&\geq \mathbf{Pr}\left(|(x_t)_i| > 2\alpha \text{ for all } i \in \Delta_{\text{add},t_j}\right) \\
&= \left[\mathbf{Pr}\left(|(x_t)_k| > 2\alpha\right)\right]^{S_{\text{add}}}, \quad k \in \Delta_{\text{add},t_j} \text{ arbitrary} \\
&= \left[2\mathcal{Q}\left(\frac{2\alpha}{\sigma_{\text{sys}}\sqrt{t - t_j + 1}}\right)\right]^{S_{\text{add}}}.
\end{aligned}$$

We examine the particular case where $t = t_j + \tau_{\text{det}}$. In this case,

$$\begin{aligned}
\mathbf{Pr}\left(\hat{N}_{t_j+\tau_{\text{det}}} = N_{t_j+\tau_{\text{det}}} \,\middle|\, \mathbf{F_j}\right) &\geq \left[2\mathcal{Q}\left(\frac{2\alpha}{\sigma_{\text{sys}}\sqrt{(t_j+\tau_{\text{det}}+1)-t_j}}\right)\right]^{S_{\text{add}}} \\
&= \left[2\mathcal{Q}\left(\frac{2\alpha}{\sigma_{\text{sys}}\sqrt{\tau_{\text{det}}+1}}\right)\right]^{S_{\text{add}}} \\
&\geq 1-\varepsilon,
\end{aligned}$$

where the final inequality is easily verified and follows from the ceiling in the definition of τ_{det} and the fact that $\mathcal{Q}$ is a decreasing function.

If $\hat{N}_t = N_t$ for $t = t_j + \tau_{\text{det}}$, then the model assumptions of no support deletions and no support additions until time t_{j+1}, in addition to the result of claim 2, imply that $\hat{N}_t = N_t$ for all $t \in [t_j + \tau_{\text{det}} : t_{j+1} - 1]$, which is exactly the event $\mathbf{E_j}$. Therefore, $\mathbf{Pr}\left(\mathbf{E_j} \,|\, \mathbf{F_j}\right) = \mathbf{Pr}\left(\hat{N}_{t_j+\tau_{\text{det}}} = N_{t_j+\tau_{\text{det}}} \,\middle|\, \mathbf{F_j}\right) \geq 1-\varepsilon$, which completes the proof. $\square$

Lemma 2. *Assume that $\{x_t\}$ and $\{y_t\}$ follow the signal model above, $\{t\}$ is a discrete set of sampling times, only additions to true support ($N_t \subseteq N_{t+1}$ for all t), etc.*

$\delta_{S_{max}}(A) < 1$, $\alpha_{del} = 0$.

Define the event $\mathbf{D} = \{\hat{N}_t = N_t = N_* \text{ for all } t \in [t_* : t_{**}]\}$, *where N_* is some fixed index set.*

At each time t, let $\hat{x}_t = \hat{x}_{t,KFCS}$ be the KFCS estimate of x_t (Algorithm 1) and let $\tilde{x}_t = \tilde{x}_{t,GAKF}$ be the GAKF estimate of x_t (Algorithm 2).

Then given any $\varepsilon > 0$ there exists some $t_{ms} \geq t_*$ such that for all $t \in [t_{ms} : t_{**}]$, we have $\mathbb{E}\left[\|\tilde{x}_t - \hat{x}_t\|_2^2 \,|\, \mathbf{D}\right] < \varepsilon$, i.e., $\hat{x}_t$ converges to $\tilde{x}_t$ in mean square.

Proof. Throughout, we assume that the event $\mathbf{D}$ occurs and $t \in [t_* : t_{**}]$.

Where possible, we consider variables and parameters only along the support set N_*, but to simplify notation will omit the subscript N_*. Thus, $\nu_t = (\nu_t)_{N_*}$, $A = A_{[1:n],N_*}$, $Q = Q_{N_*,N_*}$, $\hat{x}_t = (\hat{x}_t)_{N_*}$, $J_t = (J_t)_{N_*,N_*}$, $K_t = (K_t)_{N_*,[1:n]}$, $P_{t|t-1} = (P_{t|t-1})_{N_*,N_*}$, $P_t = (P_t)_{N_*,N_*}$, and analogously for $\tilde{x}_t$, $\tilde{J}_t$, $\tilde{K}_t$, $\tilde{P}_{t|t-1}$, and $\tilde{P}_t$.

Note, however, that y_t and w_t may be be supported on $[1 : n]$ and are thus not truncated when they appear; similarly, R is not truncated.

For $t > t_*$, both KFCS and GAKF run the same fixed-dimensional and fixed-parameter Kalman filter for $(x_t)_{N_*}$, but with different initial conditions. **Elaborate...**

$\downarrow$———— **moved to enhance the flow of the proof**

Suppose that $t \in [t_* : t_{**}]$. We see that

$$
\begin{aligned}
P_{t+1|t} &= P_t + Q \\
&= (I - K_t A) P_{t|t-1} + Q \\
&= P_{t|t-1} + Q - P_{t|t-1} A' (A P_{t|t-1} A' + R)^{-1} A P_{t|t-1},
\end{aligned}
$$

which is a discrete algebraic Riccati recursion (2) with $F = I$, $G = I$, $Q = \sigma_{\text{sys}}^2 I_{|N_*| \times |N_*|} \succ 0$, $R = \sigma_{\text{obs}}^2 I_{n \times n} \succ 0$, and $S = 0$. **Verify Q, R – goes back to the algorithm issues.** Note that Q is constant on $[t_* : t_{**}]$ since we assume that $\mathbf{D}$ occurs.

Since $|N_*| \leq S_{\max}$ and $\delta_{S_{\max}} < 1$, $A = (A_{[1:n],N_*})$ is full rank. Therefore, using the results from Section 5.1, $\{I, A\}$ is detectable. Further, since $Q = \sigma_{\text{sys}}^2 I$ is full rank, $\{I, Q\}$ is stabilizable.

Referring to the algorithm **(which one?)**, we see that $P_0 = P_{t_0 - 1} = \sigma_{\text{sys},0}^2 I \succ 0$. **is this even true? Need to get the algorithms and model set up correctly. I think we want the initial step to be $P_{t_0|t_0-1} = \sigma_{\text{sys},0}^2 I + Q_{t_0} \succeq 0$, but the two algorithms disagree on what Q_{t_0} is.**

Therefore, by [2], Lemma 8.7.3, the DARR converges to a positive semi-definite matrix $P_\star$ which satisfies the corresponding DARE. This implies that $K_t \to K_\star = P_\star A' (A P_\star A' + R)^{-1}$

and $J_t \to J_\star = (I - K_\star A)$. Further, by [2], Theorem 7.5.1.b, $\rho(J_\star) = \rho(I - K_\star A) < 1$.

Since GAKF and KFCS run the same Kalman filter, these results also apply to the GAKF iterates, i.e. $\tilde{P}_{t|t-1} \to P_\star$, $\tilde{K}_t \to K_\star$, and $\tilde{J}_t \to J_\star$.

Define $\rho = \rho(J_\star)$ and let $\varepsilon_0 = (1-\rho)/2$. A standard result from linear algebra states that there exists a matrix norm $\|\cdot\|_\rho$ such that $\|J_\star\|_\rho \le \rho + \varepsilon_0 = (1+\rho)/2 < 1$. Further, by the equivalence of matrix norms on a finite-dimensional space, there exists some constant $c_{2,\rho}$ such that $\|M\|_2 \le c_{2,\rho}\|M\|_\rho$ for any matrix M.

Since $\tilde{J}_t \to J_\star$, there exists some $t_c \ge t_0$ such that for all $t \ge t_c$, $\|\tilde{J}_t\|_2 < \|J_\star\|_2 + 1$. Therefore, for any $t \ge t_0$, we have $\|\tilde{J}_t\|_2 \le \max\{\|\tilde{J}_{t_0}\|_2, \|\tilde{J}_{t_0+1}\|_2, \ldots, \|\tilde{J}_{t_c-1}\|_2, \|J_\star\|_2 + 1\}$, i.e. there exists some value $\tilde{B}_J > 0$ such that $\|\tilde{J}_t\|_2 < \tilde{B}_J$ for all t. Since $\|\tilde{J}_t\|_2 < \infty$ for all t and $\|J_\star\|_2 < \infty$, we must also have $\tilde{B}_J < \infty$.

By similar arguments, since J_t converges to $J_\star$ and $P_{t|t-1}$ and $\tilde{P}_{t|t-1}$ converge to $P_\star$, there exist some $0 < B_J, B_P, \tilde{B}_P < \infty$ such that $\|J_t\|_2 < B_J$, $\|P_{t|t-1}\|_2 < B_P$, and $\|\tilde{P}_{t|t-1}\|_2 < \tilde{B}_P$ for all t.

Let $\varepsilon > 0$ be arbitrary.

The convergence results above and standard analysis techniques can be used to show that there exists some $t_\varepsilon > t_*$ such that for all $t \ge t_\varepsilon$, all of the following conditions hold:

- $\|K_t - \tilde{K}_t\|_2 < \varepsilon$;
- $\|J_t - \tilde{J}_t\|_2 < \varepsilon$; and
- $\|J_t\|_\rho \le \|J_\star\|_\rho + (1-\rho)/4$.

$\star$ Why do we care if t_ε does not depend on y?

Problem: NV proof says $\hat{x}_{t_*}$ is independent of y_{t_*}, but by definition it's not. AB draft: $t_* - 1$. So I agree that we're independent of $y_1 \ldots y_{t_*-1}$, but we are dependent on $y_{t_*} \ldots y_t$ because $\hat{x}_t = J_t\hat{x}_{t-1} + K_t y_t$ for $t > t_*$. **All of this independence stuff needs to be very carefully worked and verified; also, why do we care? I think $\hat{x}$ is useless here, it does not affect the choice of t_ε.**

Attempted fix: Examining the algorithms, we see that K_t, $\tilde{K}_t$, J_t, $\tilde{J}_t$, $P_{t|t-1}$ and $\tilde{P}_{t|t-1}$ do not depend on $\{y_k\}$, hence, neither do $K_\star$, $J_\star$, and $P_\star$. It follows that t_ε also does not depend on $\{y_k\}$.

↑———— /moved

Let $\hat{e}_t = x_t - \hat{x}_t$ and $\tilde{e}_t = x_t - \tilde{x}_t$. Define $\mathrm{diff}_t = \hat{e}_t - \tilde{e}_t$ and notice that $\mathrm{diff}_t = \tilde{x}_t - \hat{x}_t$.

Let $t > t_\varepsilon > t_*$. By Algorithm 1 and the model, we see that

$$\begin{aligned}
\hat{e}_t &= x_t - \hat{x}_t \\
&= (x_{t-1} + \nu_t) - (J_t \hat{x}_{t-1} + K_t y_t) \\
&= x_{t-1} + \nu_t - J_t \hat{x}_{t-1} - K_t (A x_t + w_t) \\
&= x_{t-1} + \nu_t - J_t \hat{x}_{t-1} - K_t A (x_{t-1} + \nu_t) - K_t w_t \\
&= (I - K_t A) x_{t-1} - J_t \hat{x}_{t-1} + (I - K_t A) \nu_t - K_t w_t \\
&= J_t (x_{t-1} - \hat{x}_{t-1}) + J_t \nu_t - K_t w_t \\
&= J_t \hat{e}_{t-1} + J_t \nu_t - K_t w_t.
\end{aligned}$$

Similarly, using Algorithm 2 and the model, we can verify that

$$\tilde{e}_t = \tilde{J}_t \tilde{e}_{t-1} + \tilde{J}_t \nu_t - \tilde{K}_t w_t.$$

Combining these results yields

$$\mathrm{diff}_t = J_t \mathrm{diff}_{t-1} + (J_t - \tilde{J}_t)(\tilde{e}_{t-1} + \nu_t) + (\tilde{K}_t - K_t) w_t.$$

Let

$$u_t = (J_t - \tilde{J}_t)(\tilde{e}_{t-1} + \nu_t) + (\tilde{K}_t - K_t) w_t,$$

so that $\mathrm{diff}_t = J_t \mathrm{diff}_{t-1} + u_t$. Recursively applying this identity, we see that

$$\begin{aligned}
\mathrm{diff}_t &= J_t \mathrm{diff}_{t-1} + u_t \\
&= J_t \left(J_{t-1} \mathrm{diff}_{t-2} + u_{t-1} \right) + u_t \\
&= J_t J_{t-1} \mathrm{diff}_{t-2} + J_t u_{t-1} + u_t \\
&= J_t J_{t-1} \left(J_{t-2} \mathrm{diff}_{t-3} + u_{t-2} \right) + J_t u_{t-1} + u_t \\
&= J_t J_{t-1} J_{t-2} \mathrm{diff}_{t-3} + J_t J_{t-1} u_{t-2} + J_t u_{t-1} + u_t \\
&\;\vdots \\
&= J_t J_{t-1} \cdots J_{t_\varepsilon+1} \mathrm{diff}_{t_\varepsilon} + J_t J_{t-1} \cdots J_{t_\varepsilon+2} u_{t_\varepsilon+1} + \ldots + J_t u_{t-1} + u_t.
\end{aligned}$$

If we define

$$M_k^t = \begin{cases} J_t J_{t-1} \cdots J_{k+1} J_k & k \le t \\ I & k > t \end{cases}$$

then we can more compactly write

$$\mathrm{diff}_t = M_{t_\varepsilon+1}^t \mathrm{diff}_{t_\varepsilon} + \sum_{k=t_\varepsilon+1}^{t} M_{k+1}^t u_k.$$

Therefore, applying the triangle and Cauchy-Schwarz inequalities for expectation and noting that the matrices $\{M_k^t\}$ are deterministic,

$$
\begin{aligned}
\mathbb{E}\left[\|\mathrm{diff}_t\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} &= \mathbb{E}\left[\left\|M_{t_\varepsilon+1}^t \mathrm{diff}_{t_\varepsilon} + \sum_{k=t_\varepsilon+1}^{t} M_{k+1}^t u_k\right\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} \\
&\leq \mathbb{E}\left[\left\|M_{t_\varepsilon+1}^t \mathrm{diff}_{t_\varepsilon}\right\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} + \mathbb{E}\left[\|u_t\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} + \sum_{k=t_\varepsilon+1}^{t-1} \mathbb{E}\left[\left\|M_{k+1}^t u_k\right\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} \\
&\leq \mathbb{E}\left[\left\|M_{t_\varepsilon+1}^t\right\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} \mathbb{E}\left[\|\mathrm{diff}_{t_\varepsilon}\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} + \\
&\qquad \mathbb{E}\left[\|u_t\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} + \sum_{k=t_\varepsilon+1}^{t-1} \mathbb{E}\left[\left\|M_{k+1}^t\right\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} \mathbb{E}\left[\|u_k\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} \\
\mathbb{E}\left[\|\mathrm{diff}_t\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} &\leq \|M_{t_\varepsilon+1}^t\|_2 \, \mathbb{E}\left[\|\mathrm{diff}_{t_\varepsilon}\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} + \\
&\qquad \left(1 + \sum_{\ell=t_\varepsilon+2}^{t} \|M_\ell^t\|_2\right) \max_{\tau \in [t_\varepsilon+1:t]} \left\{\mathbb{E}\left[\|u_\tau\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}}\right\}. \qquad (3)
\end{aligned}
$$

Recall that, for $k \geq t_\varepsilon$, we have

$$\|J_k\|_\rho \leq \|J_\star\|_\rho + (1-\rho)/4 \leq (1+\rho)/2 + (1-\rho)/4 = (3+\rho)/4 < 1.$$

Let $a = (3+\rho)/4$. Then for $t_\varepsilon \leq k \leq t$,

$$
\begin{aligned}
\|M_k^t\|_2 &\leq c_{2,\rho} \|M_k^t\|_\rho \\
&= \|J_t J_{t-1} \cdots J_k\|_\rho \\
&\leq \|J_t\|_\rho \|J_{t-1}\|_\rho \cdots \|J_k\|_\rho \\
\|M_k^t\|_2 &\leq c_{2,\rho} a^{t-k+1}. \qquad (4)
\end{aligned}
$$

With this, we see that

$$
\begin{aligned}
\left(1 + \sum_{\ell=t_\varepsilon+2}^{t} \|M_\ell^t\|_2\right) &\leq \left(1 + \sum_{\ell=t_\varepsilon+2}^{t} c_{2,\rho} a^{t-\ell+1}\right) \\
&\leq \max\{1, c_{2,\rho}\} \cdot \sum_{\ell=0}^{\infty} a^\ell \\
\left(1 + \sum_{\ell=t_\varepsilon+2}^{t} \|M_\ell^t\|_2\right) &\leq \max\{1, c_{2,\rho}\} \cdot \frac{1}{1-a} \ . \qquad (5)
\end{aligned}
$$

Let $\tau \in [t_\varepsilon + 1 : t]$ be arbitrary. Since $\tau > t_\varepsilon$, we have $\|\tilde{K}_\tau - K_\tau\|_2 < \varepsilon$ and $\|\tilde{J}_\tau - J_\tau\|_2 < \varepsilon$.

Consider

$$
\begin{aligned}
\mathbb{E}\left[\|u_\tau\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} &= \mathbb{E}\left[\left\|(J_\tau - \tilde{J}_\tau)(\tilde{e}_{\tau-1} + \nu_\tau) + (\tilde{K}_\tau - K_\tau)w_\tau\right\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} \\
&\leq \|J_\tau - \tilde{J}_\tau\|_2 \, \mathbb{E}\left[\|\tilde{e}_{\tau-1} + \nu_\tau\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} + \|\tilde{K}_\tau - K_\tau\|_2 \, \mathbb{E}\left[\|w_\tau\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} \\
&< \varepsilon \cdot \mathbb{E}\left[\|\tilde{e}_{\tau-1} + \nu_\tau\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} + \varepsilon \cdot \mathbb{E}\left[\|w_\tau\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} \\
\mathbb{E}\left[\|u_\tau\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} &\leq \varepsilon \left(\mathbb{E}\left[\|\tilde{e}_{\tau-1}\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} + \mathbb{E}\left[\|\nu_\tau\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} + \mathbb{E}\left[\|w_\tau\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}}\right), \qquad (6)
\end{aligned}
$$

where we have used the triangle and Cauchy-Schwarz inequalities for expectation.

By the properties of the Kalman filter, for any k, we have

$$
\begin{aligned}
\tilde{P}_k &= \mathbb{E}\left[(\tilde{x}_k - \mathbb{E}[\tilde{x}_k \,|\, y_1, y_2, \ldots, y_k])(\tilde{x}_k - \mathbb{E}[\tilde{x}_k \,|\, y_1, y_2, \ldots, y_k])' \,\middle|\, y_1, y_2, \ldots, y_k\right] \\
&= \mathbb{E}\left[(\tilde{x}_k - x_k)(\tilde{x}_k - x_k)' \,\middle|\, y_1, y_2, \ldots, y_k\right] \\
&= \mathbb{E}\left[\tilde{e}_k \tilde{e}_k' \,|\, y_1, y_2, \ldots, y_k\right] \\
&= \mathbb{E}\left[\tilde{e}_k \tilde{e}_k'\right],
\end{aligned}
$$

where the independence on the last line follows because $\tilde{P}_k$ has no dependence on any of the $\{y_i\}$, a well-known property of the Kalman filter (and consequence of the algorithm). Therefore,

$$
\begin{aligned}
\mathbb{E}\left[\|\tilde{e}_k\|_2^2 \,\middle|\, \mathbf{D}\right] &= \mathrm{tr}\left(\mathbb{E}\left[\tilde{e}_k' \tilde{e}_k \,|\, \mathbf{D}\right]\right) \\
&= \mathbb{E}\left[\mathrm{tr}(\tilde{e}_k' \tilde{e}_k) \,|\, \mathbf{D}\right] \\
&= \mathbb{E}\left[\mathrm{tr}(\tilde{e}_k \tilde{e}_k') \,|\, \mathbf{D}\right] \\
&= \mathrm{tr}\left(\mathbb{E}\left[\tilde{e}_k \tilde{e}_k' \,|\, \mathbf{D}\right]\right) \\
&= \mathrm{tr}\left(\mathbb{E}\left[\tilde{e}_k \tilde{e}_k'\right]\right) \\
&= \mathrm{tr}\left(\tilde{P}_k\right).
\end{aligned}
$$

Here, we used the fact that the occurrence of $\mathbf{D}$ is independent of the value of $\tilde{P}_k = \mathbb{E}[e_k e_k']$. **Make sure this is legitimate.**

We see that

$$\|\tilde{P}_k\|_2 = \|\tilde{J}_k \tilde{P}_{k|k-1}\|_2 \leq \|\tilde{J}_k\|_2 \|\tilde{P}_{k|k-1}\|_2 < \tilde{B}_J \tilde{B}_P < \infty,$$

where we recall that $\|\tilde{P}_{k|k-1}\|_2 < \tilde{B}_P$ and $\|\tilde{J}_k\|_2 < \tilde{B}_J$. Since $\tilde{P}_k$ is Hermitian, $\|\tilde{P}_k\|_2 = \lambda_{\max}(\tilde{P}_k)$. Therefore,

$$\mathrm{tr}(\tilde{P}_k) = \sum_i \lambda_i(\tilde{P}_k) \leq |N_*| \lambda_{\max}(\tilde{P}_k) = |N_*| \|\tilde{P}_k\|_2 < |N_*| \tilde{B}_J \tilde{B}_P < \infty,$$

so there exists some $0 < \tilde{B} < \infty$ such that $\mathrm{tr}(\tilde{P}_k) < \tilde{B}$ for all k.

Therefore,

$$\mathbb{E}\left[\|\tilde{e}_{\tau-1}\|_2^2 \,\middle|\, \mathbf{D}\right] = \operatorname{tr}\left(\tilde{P}_{\tau-1}\right) < \tilde{B}.$$

Since $\mathbf{D}$ occurs, ν_τ is supported on N_*, so the covariance of $\nu_\tau = (\nu_\tau)_{N_*}$ is $\mathbb{E}[\nu_\tau \nu_\tau'] = \mathbb{E}\left[\nu_\tau \nu_\tau' \,\middle|\, \mathbf{D}\right] = \sigma_{\text{sys}}^2 I_{|N_*|\times|N_*|}$. **VERIFY this claim.** Therefore,

$$\begin{aligned}
\mathbb{E}\left[\|\nu_\tau\|_2^2 \,\middle|\, \mathbf{D}\right] &= \operatorname{tr}\left(\mathbb{E}\left[\nu_\tau' \nu_\tau \,\middle|\, \mathbf{D}\right]\right) \\
&= \mathbb{E}\left[\operatorname{tr}(\nu_\tau' \nu_\tau) \,\middle|\, \mathbf{D}\right] \\
&= \mathbb{E}\left[\operatorname{tr}(\nu_\tau \nu_\tau') \,\middle|\, \mathbf{D}\right] \\
&= \operatorname{tr}\left(\mathbb{E}\left[\nu_\tau \nu_\tau' \,\middle|\, \mathbf{D}\right]\right) \\
&= \operatorname{tr}\left(\sigma_{\text{sys}}^2 I_{|N_*|\times|N_*|}\right) \\
&= |N_*|\sigma_{\text{sys}}^2.
\end{aligned}$$

A similar computation proves that $\mathbb{E}\left[\|w_\tau\|_2^2 \,\middle|\, \mathbf{D}\right] = n\sigma_{\text{obs}}^2$.

With (6), these results show that

$$\mathbb{E}\left[\|u_\tau\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} < \varepsilon\left(\sqrt{\tilde{B}} + \sqrt{|N_*|\sigma_{\text{sys}}^2} + \sqrt{n\sigma_{\text{obs}}^2}\right).$$

Since $\tau \in [t_\varepsilon + 1 : t]$ was arbitrary, we conclude that

$$\max_{\tau\in[t_\varepsilon+1:t]}\left\{\mathbb{E}\left[\|u_\tau\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}}\right\} < \varepsilon\left(\sqrt{\tilde{B}} + \sqrt{|N_*|\sigma_{\text{sys}}^2} + \sqrt{n\sigma_{\text{obs}}^2}\right). \tag{7}$$

We have seen that $\mathbb{E}\left[\|\tilde{e}_k\|_2^2 \,\middle|\, \mathbf{D}\right] < \operatorname{tr}(\tilde{P}_k) < \tilde{B}$ for some $\tilde{B}$ and all k; by similar work, we can conclude that there exists some B such that $\mathbb{E}\left[\|e_k\|_2^2 \,\middle|\, \mathbf{D}\right] < \operatorname{tr}(P_k) < B$ for all k. Therefore, by the triangle inequality for expectation,

$$\begin{aligned}
\mathbb{E}\left[\|\text{diff}_{t_\varepsilon}\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} &= \mathbb{E}\left[\|e_{t_\varepsilon} - \tilde{e}_{t_\varepsilon}\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} \\
&\le \mathbb{E}\left[\|e_{t_\varepsilon}\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} + \mathbb{E}\left[\|\tilde{e}_{t_\varepsilon}\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} \\
\mathbb{E}\left[\|\text{diff}_{t_\varepsilon}\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} &< B + \tilde{B}. \qquad (8)
\end{aligned}$$

Combining (3) with (4), (5), (7), and (8), we see that

$$\mathbb{E}\left[\|\text{diff}_t\|_2^2 \,\middle|\, \mathbf{D}\right]^{\frac{1}{2}} < c_{2,\rho} a^{t-t_\varepsilon}(B + \tilde{B}) + C\varepsilon,$$

where $C = \max\{1, c_{2,\rho}\} \cdot \frac{1}{1-a} \cdot \left(\sqrt{\tilde{B}} + \sqrt{|N_*|\sigma_{\text{sys}}^2} + \sqrt{n\sigma_{\text{obs}}^2}\right)$.

If

$$t_{\mathrm{ms}} = t_\varepsilon + \left\lceil \log_a \left(\frac{C\varepsilon}{c_{2,\rho}(B + \tilde{B})} \right) \right\rceil,$$

then we see that for all $t \geq t_{\mathrm{ms}}$,

$$\mathbb{E}\left[\|\tilde{x}_t - \hat{x}_t\|_2^2 \,\middle|\, \mathbf{D} \right]^{\frac{1}{2}} = \mathbb{E}\left[\|\mathrm{diff}_t\|_2^2 \,\middle|\, \mathbf{D} \right]^{\frac{1}{2}} < 2C\varepsilon,$$

and since C is constant and ε is arbitrary we have obtained our desired result. $\square$

Corollary 1. *Assume that the conditions of Lemma 2 hold.*

Then given any ε and ε_{err} there exists some $\tau_{KF} = \tau_{KF}(\varepsilon, \varepsilon_{err}, N_*)$ such that for all $t \in [t_* + \tau_{KF} : t_{**}]$, we have $\boldsymbol{Pr}\left(\|\tilde{x}_t - \hat{x}_t\|_2^2 \leq \varepsilon_{err} \,\middle|\, \mathbf{D}\right) > 1 - \varepsilon$. Note that if $t_* + \tau_{KF} > t_{**}$, then this interval is empty and the result is vacuously true.

Proof. Let $\varepsilon > 0$ and $\varepsilon_{\mathrm{err}} > 0$ be given and let $\tilde{\varepsilon} \leq \varepsilon \cdot \varepsilon_{\mathrm{err}}$. By Lemma 2, there exists some $t_{\mathrm{ms}} = t_{\mathrm{ms}}(\tilde{\varepsilon}, N_*)$ such that for all $t \geq t_{\mathrm{ms}}$,

$$\mathbb{E}\left[\|\tilde{x}_t - \hat{x}_t\|_2^2 \,\middle|\, \mathbf{D}\right] < \tilde{\varepsilon} \leq \varepsilon \cdot \varepsilon_{\mathrm{err}}.$$

Let $t \geq t_{\mathrm{ms}}$. By Markov's inequality,

$$\mathbf{Pr}\left(\|\tilde{x}_t - \hat{x}_t\|_2^2 > \varepsilon_{\mathrm{err}} \,\middle|\, \mathbf{D}\right) \leq \frac{\mathbb{E}\left[\|\tilde{x}_t - \hat{x}_t\|_2^2 \mid \mathbf{D}\right]}{\varepsilon_{\mathrm{err}}} < \frac{\tilde{\varepsilon}}{\varepsilon_{\mathrm{err}}} \leq \varepsilon.$$

Define $\tau_{\mathrm{KF}} = t_{\mathrm{ms}} - t_*$. Since t_{ms} is a function of $\tilde{\varepsilon}$, which is itself a function of ε and $\varepsilon_{\mathrm{err}}$, we have $\tau_{\mathrm{KF}} = \tau_{\mathrm{KF}}(\varepsilon, \varepsilon_{\mathrm{err}}, N_*)$, and for all $t \geq t_{\mathrm{ms}} = t_* + \tau_{\mathrm{KF}}$,

$$\mathbf{Pr}\left(\|\tilde{x}_t - \hat{x}_t\|_2^2 < \varepsilon_{\mathrm{err}} \,\middle|\, \mathbf{D}\right) > 1 - \varepsilon,$$

which is our desired result. $\square$

Theorem 1. *Assume that the conditions of Lemma 1 and Lemma 2 hold.* ***Recall def of*** E_j ***and*** F_j***?***

Let $\varepsilon > 0$, $\varepsilon_{err} > 0$ *be given.*

Let $\tau_{det} = \tau_{det}(\alpha, \varepsilon)$ *be as in Lemma 1.*

Choose $d > \tau_{det} + \max_j\{\tau_{KF}(\varepsilon, \varepsilon_{err}, N_{t_j})\}$.

1) *Given any* $j \in [0 : K-1]$, $\boldsymbol{Pr}(\|\tilde{x}_t - \hat{x}_t\|_2^2 \le \varepsilon_{err}) > (1-\varepsilon)^{j+2}$ *for all* $t \in [t_j + \tau_{det} + \tau_{KF}(\varepsilon, \varepsilon_{err}, N_{t_j}) : t_{j+1} - 1]$.
2) $\boldsymbol{Pr}(|\Delta_t| \le S_{add}$ *and* $|\Delta_e| = 0$ *for all* $t \ge t_0) \ge (1-\varepsilon)^K$
3) $\boldsymbol{Pr}($*For all* $j \in [0 : K-1]$, $|\Delta_t| = 0$ *and* $|\Delta_e| = 0$ *for all* $t \in [t_j + \tau_{det} : t_{j+1} - 1])$ $\ge (1-\varepsilon)^K$

Proof. **See also comments at the end of this section for an alternate derivation.**

We first show by induction that $\mathbf{Pr}(\mathbf{E_j}) \ge (1-\varepsilon)^{j+1}$ for all $j \ge 0$.

Consider the base case, where $j = 0$. By assumption, $\hat{N}_{t_0-1} = N_{t_0-1}$, so $\mathbf{F_0}$ occurs. We have

$$\mathbf{Pr}(\mathbf{E_0}) = \mathbf{Pr}\left(\mathbf{E_0} \,|\, \mathbf{F_0}\right) \ge 1 - \varepsilon$$

by Lemma 1, which proves the base case.

Now assume that the claim is true for $j = (k-1)$ for some $k \ge 1$, that is, $\mathbf{Pr}(\mathbf{E_{k-1}}) \ge (1-\varepsilon)^k$. Consider

$$\begin{aligned}
\mathbf{Pr}(\mathbf{E_k}) &= \mathbf{Pr}(\mathbf{E_k} \cap \mathbf{E_{k-1}}) + \mathbf{Pr}(\mathbf{E_k} \cap (\mathbf{E_{k-1}})^c) \\
&\ge \mathbf{Pr}(\mathbf{E_k} \cap \mathbf{E_{k-1}}) \\
&= \mathbf{Pr}\left(\mathbf{E_k} \,|\, \mathbf{E_{k-1}}\right) \mathbf{Pr}(\mathbf{E_{k-1}}) \\
&= \mathbf{Pr}\left(\mathbf{E_k} \,|\, \mathbf{F_k}\right) \mathbf{Pr}(\mathbf{E_{k-1}}) \qquad \textbf{WHY is this true?} \\
&\ge (1-\varepsilon)(1-\varepsilon)^k \\
&= (1-\varepsilon)^{k+1},
\end{aligned}$$

where we applied Lemma 1 to conclude that $\mathbf{Pr}\left(\mathbf{E_k} \,|\, \mathbf{F_k}\right) \ge 1-\varepsilon$. Therefore, by the principle of mathematical induction, we conclude that

$$\mathbf{Pr}(\mathbf{E_j}) \ge (1-\varepsilon)^{j+1}$$

for all $j \ge 0$.

Fix $j \in [0 : K-1]$.

Choosing $t_* = t_j + \tau_{\text{det}}$ and $t_{**} = t_{j+1} - 1$, the event $\mathbf{D} = \{\hat{N}_t = N_t = N_* \text{ for all } t \in [t_* : t_{**}]\}$ is identically the event $\mathbf{E_j} = \{\hat{N}_t = N_t \text{ for all } t \in [t_j + \tau_{\text{det}} : t_{j+1} - 1]\}$. Corollary 1 thus yields

$$\mathbf{Pr}\left(\|\tilde{x}_t - \hat{x}_t\|_2^2 \le \varepsilon_{\text{err}} \,\middle|\, \mathbf{E_j}\right) > 1 - \varepsilon$$

for all $t \in [t_j + \tau_{\text{det}} + \tau_{\text{KF}}(\varepsilon, \varepsilon_{\text{err}}, N_{t_j}) : t_{j+1} - 1]$.

Note that since $d > \tau_{\text{det}} + \tau_{\text{KF}}(\varepsilon, \varepsilon_{\text{err}}, N_{t_j})$ for all j, the interval $[t_j + \tau_{\text{det}} + \tau_{\text{KF}}(\varepsilon, \varepsilon_{\text{err}}, N_{t_j}) : t_{j+1} - 1]$ is nonempty.

For any $t \in [t_j + \tau_{\text{det}} + \tau_{\text{KF}} : t_{j+1} - 1]$, we see that

$$\begin{aligned}
\mathbf{Pr}\left(\|\tilde{x}_t - \hat{x}_t\|_2^2 \leq \varepsilon_{\text{err}}\right) &= \mathbf{Pr}\left(\{\|\tilde{x}_t - \hat{x}_t\|_2^2 \leq \varepsilon_{\text{err}}\} \cap \mathbf{E_j}\right) + \mathbf{Pr}\left(\{\|\tilde{x}_t - \hat{x}_t\|_2^2 \leq \varepsilon_{\text{err}}\} \cap (\mathbf{E_j})^c\right) \\
&\geq \mathbf{Pr}\left(\{\|\tilde{x}_t - \hat{x}_t\|_2^2 \leq \varepsilon_{\text{err}}\} \cap \mathbf{E_j}\right) \\
&= \mathbf{Pr}\left(\|\tilde{x}_t - \hat{x}_t\|_2^2 \leq \varepsilon_{\text{err}} \,\middle|\, \mathbf{E_j}\right) \mathbf{Pr}(\mathbf{E_j}) \\
&> (1-\varepsilon)(1-\varepsilon)^{j+1} \\
&= (1-\varepsilon)^{j+2},
\end{aligned}$$

which verifies **the first claim.**

I think that the third claim's probability equals the one below. Either way, we need this.

$$\begin{aligned}
\mathbf{Pr}(\mathbf{E_0} \cap \mathbf{E_1} \cap \ldots \cap \mathbf{E_{K-1}}) &= \mathbf{Pr}(\mathbf{E_0})\mathbf{Pr}\left(\mathbf{E_1} \,|\, \mathbf{E_0}\right)\mathbf{Pr}\left(\mathbf{E_2} \,|\, \mathbf{E_0} \cap \mathbf{E_1}\right) \cdots \mathbf{Pr}\left(\mathbf{E_{K-1}} \,\middle|\, \bigcap_{j=0}^{K-1} \mathbf{E_j}\right) \\
&= \mathbf{Pr}(\mathbf{E_0})\mathbf{Pr}\left(\mathbf{E_1} \,|\, \mathbf{E_0}\right)\mathbf{Pr}\left(\mathbf{E_2} \,|\, \mathbf{E_1}\right) \cdots \mathbf{Pr}\left(\mathbf{E_{K-1}} \,|\, \mathbf{E_{K-2}}\right) \\
&= \mathbf{Pr}(\mathbf{E_0})\mathbf{Pr}\left(\mathbf{E_1} \,|\, \mathbf{F_1}\right)\mathbf{Pr}\left(\mathbf{E_2} \,|\, \mathbf{F_2}\right) \cdots \mathbf{Pr}\left(\mathbf{E_{K-1}} \,|\, \mathbf{F_{K-1}}\right) \\
&\geq (1-\varepsilon)^K
\end{aligned}$$

Second claim: the event seems to be a superset of the event $E_0 \cap \ldots \cap E_{K-1}$, so obviously the probability is bigger than $(1-\varepsilon)^K$ using the result above.

Stuff to verify:

$\mathbf{Pr}\left(E_j \,|\, E_{j-1}\right) = \mathbf{Pr}\left(E_j \,|\, F_j\right)$: if E_{j-1} happens, then F_j definitely happens, but not seeing why these are equal yet.

Markov property used on $\{E_j\}$: justification

$\star\star\star$ **See next page for proposed fix / work.**

Proposed work to get around all of these issues (very rough):

From Lemma 1, we have $\hat{N}_t \subseteq N_t$ for all t, and the model assumptions yield $N_{t-1} \subseteq N_t$ for all t. Therefore,

$$\mathbf{E_j} = \{\hat{N}_t = N_t \text{ for all } t \in [t_j + \tau_{\text{det}} : t_{j+1} - 1]\} = \{\hat{N}_{t_j+\tau_{\text{det}}} = N_{t_j+\tau_{\text{det}}}\}.$$

Also from Lemma 1,

$$\mathbf{Pr}\left(|(x_{t_j+\tau_{\text{det}}})_i| > 2\alpha \text{ for all } i \in \Delta_{\text{add},t_j}\right) \geq 1 - \varepsilon.$$

From Lemma 1, if the event $\mathbf{F_j} = \{\hat{N}_{t_j-1} = N_{t_j-1}\}$ occurs, then $\Delta_t \subseteq \Delta_{\text{add},t_j}$ for all $t \in [t_j : t_{j+1} - 1]$.

Notice that $\mathbf{F_j} \subseteq \mathbf{E_{j-1}}$, i.e. if $\mathbf{E_{j-1}}$ occurs, then $\mathbf{F_j}$ occurs. Similarly, $\mathbf{F_j} \subseteq \bigcap_{k=0}^{j-1} \mathbf{E_k}$. Therefore, if any of these events occur, then $\Delta_t \subseteq \Delta_{\text{add},t_j}$ for all $t \in [t_j : t_{j+1} - 1]$.

Now,

$$\mathbf{E_j} = \{\hat{N}_{t_j+\tau_{\text{det}}} = N_{t_j+\tau_{\text{det}}}\} = \{|(\hat{x}_{t_j+\tau_{\text{det}},\text{CSres}})_k| > \alpha \text{ for all } k \in \Delta_{t_j+\tau_{\text{det}}}\}.$$

Conditioning $\mathbf{E_j}$ on any of $\mathbf{F_j}$, $\mathbf{E_{j-1}}$, or $\bigcap_{k=0}^{j-1} \mathbf{E_k}$ forces $\Delta_{t_j+\tau_{\text{det}}} \subseteq \Delta_{\text{add},t_j}$.

We have seen that if $|(x_{t_j+\tau_{\text{det}}})_i| > 2\alpha$ for any i, then $|(\hat{x}_{t_j+\tau_{\text{det}},\text{CSres}})_i| > \alpha$, so $i \in \hat{N}_{t_j+\tau_{\text{det}}}$.

Therefore, if $|(x_{t_j+\tau_{\text{det}}})_i| > 2\alpha$ for all $i \in \Delta_{\text{add},t_j}$, then $|(\hat{x}_{t_j+\tau_{\text{det}},\text{CSres}})_i| > \alpha$ for all $i \in \Delta_{t_j+\tau_{\text{det}}}$, so $\hat{N}_{t_j+\tau_{\text{det}}} = N_{t_j+\tau_{\text{det}}}$, i.e. $\mathbf{E_j}$ occurs.

We conclude that (for any j)

$$\mathbf{Pr}\left(\mathbf{E_j} \,|\, \mathbf{F_j}\right) \geq 1 - \varepsilon$$

$$\mathbf{Pr}\left(\mathbf{E_j} \,|\, \mathbf{E_{j-1}}\right) \geq 1 - \varepsilon$$

$$\mathbf{Pr}\left(\mathbf{E_j} \,\middle|\, \bigcap_{k=0}^{j-1} \mathbf{E_k}\right) \geq 1 - \varepsilon$$

With this, we can correct the induction proof which yields $\mathbf{Pr}(\mathbf{E_j}) \geq (1-\varepsilon)^{j+1}$. Then claim 1 follows.

Claim 2 still appears to be a subset of claim 3.

Claim 3's proof is the same, except that we now can immediately jump straight to $(1-\varepsilon)^K$ after the initial application of the chain rule. In fact, we can more compactly write the proof as

$$\mathbf{Pr}\left(\bigcap_{k=0}^{K-1} \mathbf{E_j}\right) = \mathbf{Pr}(\mathbf{E_0}) \cdot \prod_{j=1}^{K-1} \mathbf{Pr}\left(\mathbf{E_j} \,\middle|\, \bigcap_{k=0}^{j-1} \mathbf{E_k}\right) \geq (1-\varepsilon) \cdot (1-\varepsilon)^{K-1}.$$

This would complete the proof.

However, this method semi-invalidates / repeats a lot of Lemma 1 part 3's proof, which implies that we can either get rid of Lemma 1 part 3 or rework Lemma 1 part 3's proof to only include relevant information.

Equivalently, if this line of logic works out, we can basically say "by the same arguments as those in Lemma 1" and only highlight the relevant piece, namely that conditioned on any of the events we're interested in, we have $\Delta_{t_j+\tau_{det}} \subseteq \Delta_{add,t_j}$, and that lets us say that $|x_t| > 2\alpha$ on Δ_{add,t_j} is good enough to bound below by $(1-\varepsilon)$.

□

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