Detecting Parametric Signals in Noise Having Exactly Known Pdf/Pmf

Reading:

- Ch. 5 in Kay-II.

- (Part of) Ch. III.B in Poor.

Detecting Parametric Signals in Noise w Having Exactly Known Pdf/Pmf $p_{\boldsymbol{w}}(\boldsymbol{w})$ (Bayesian Decision-theoretic Approach)

Consider the simple binary signal-detection problem:

$$\mathcal{H}_0: \qquad \boldsymbol{x} = \boldsymbol{\mu}_0(\varphi) + \boldsymbol{w} \quad \text{versus}$$

$$\mathcal{H}_1: \qquad \boldsymbol{x} = \boldsymbol{\mu}_1(\varphi) + \boldsymbol{w}$$

where

- $\boldsymbol{\mu}_0(\varphi)$ and $\boldsymbol{\mu}_1(\varphi)$ are *known* vector-valued functions of the nuisance parameter φ and

- the noise probability density or mass function (pdf/pmf) $p_{\boldsymbol{w}}(\boldsymbol{w})$ is *exactly known*.

Recall our discussion on handling nuisance parameters on pp. 17–18 of handout # 5. Since we have simple hypotheses, we need to specify the Bernoulli prior pmf for the two hypotheses, using prior probabilities

$$\pi_0, \quad \pi_1 = 1 - \pi_0 \quad \text{(the Bernoulli pmf)}.$$

Specializing the result in eq. (15) in handout # 5 (where we have assumed that the hypotheses and φ are independent *a*

priori, see eq. (14) in handout # 5) to the above scenario, we obtain the following Bayes' decision rule:

$$\Lambda(\boldsymbol{x}) = \frac{\int p_{\boldsymbol{w}}\big(\boldsymbol{x} - \boldsymbol{\mu}_1(\varphi)\big) \overbrace{\pi(\varphi)}^{\text{prior pdf of } \varphi} d\varphi}{\underbrace{\int p_{\boldsymbol{w}}\big(\boldsymbol{x} - \boldsymbol{\mu}_0(\varphi)\big)\, \pi(\varphi)\, d\varphi}_{\text{integrated likelihood ratio}}} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \frac{\pi_0\, \mathrm{L}(1\,|\,0)}{\pi_1\, \mathrm{L}(0\,|\,1)}.$$

(1)

Example. Detection of on-off keying signals with unknown phase in additive white Gaussian noise (AWGN): Choose AWGN with $\Sigma = \sigma^2\, I$ and known noise variance σ^2, $\boldsymbol{\mu}_0(\varphi) = \boldsymbol{0}$, and

$$\boldsymbol{\mu}_1(\varphi) = \boldsymbol{s}(\varphi) = \begin{bmatrix} s[0, \varphi] \\ s[1, \varphi] \\ \vdots \\ s[N-1, \varphi] \end{bmatrix}$$

$$= \begin{bmatrix} a_0 \sin(0 \cdot \omega_{\mathrm{c}} + \varphi) \\ a_1 \sin(1 \cdot \omega_{\mathrm{c}} + \varphi) \\ \vdots \\ a_{N-1} \sin((N-1) \cdot \omega_{\mathrm{c}} + \varphi) \end{bmatrix}$$

where $a_1, a_2, \ldots, a_N$ is a known amplitude sequence, ω_{c} is known carrier frequency, and φ is an unknown phase angle, independent of the noise, following $\pi(\varphi) = \mathrm{uniform}(0, 2\pi)$.

Now, (1) reduces to

$$
\Lambda(\boldsymbol{x}) = \frac{p(\boldsymbol{x} \mid \mathcal{H}_1)}{p(\boldsymbol{x} \mid \mathcal{H}_0)}
$$

$$
= \frac{\int_0^{2\pi} \frac{1}{2\pi}\, p(\boldsymbol{x} \mid \mathcal{H}_1, \varphi)\, d\varphi}{\int_0^{2\pi} \frac{1}{2\pi}\, \underbrace{p(\boldsymbol{x} \mid \mathcal{H}_0, \varphi)}_{\text{\color{red}indep. of } \varphi}\, d\varphi}
$$

$$
= \frac{\int_0^{2\pi} \frac{1}{2\pi} \exp\{-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1}(x[n] - s[n,\varphi])^2\}\, d\varphi}{\exp\{-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1}(x[n])^2\}}
$$

$$
= \frac{1}{2\pi} \int_0^{2\pi} \exp\left\{ \frac{1}{\sigma^2}\left[\left(\sum_{n=0}^{N-1} x[n]s[n,\varphi]\right) - \tfrac{1}{2}\sum_{n=0}^{N-1}(s[n,\varphi])^2 \right] \right\} d\varphi.
$$

We prefer to choose ω_{c} equal to an integer multiple of $2\pi/N$. For dense signal sampling (N large) and ω_{c} *not close* to 0 or π, we have [see eq. (III.B.67) in Poor]:

$$
\sum_{n=0}^{N-1}(s[n,\varphi])^2 = \sum_{n=0}^{N-1} a_n^2 \sin^2(n \cdot \omega_{\mathrm{c}} + \varphi) \approx \tfrac{1}{2}\sum_{n=0}^{N-1} a_n^2 = \frac{N}{2}\,\overline{a^2}
$$

where we have used the identity $\sin^2 x = \tfrac{1}{2} - \tfrac{1}{2}\cos 2x$ and the following definition:

$$
\overline{a^2} \;\overset{\triangle}{=}\; \frac{1}{N} \cdot \sum_{n=0}^{N-1} a_n^2.
$$

Thus,

$$\Lambda(\boldsymbol{x}) \approx \frac{1}{2\pi} \int_0^{2\pi} \exp\left\{ \frac{1}{\sigma^2} \left[\left(\sum_{n=0}^{N-1} x[n]s[n,\varphi] \right) - \tfrac{1}{4} N \overline{a^2} \right] \right\} d\varphi$$

$$= \frac{\exp\left(-\tfrac{1}{4} \frac{N \overline{a^2}}{\sigma^2} \right)}{2\pi} \cdot \int_0^{2\pi} \exp\left\{ \frac{1}{\sigma^2} \left(\sum_{n=0}^{N-1} x[n]s[n,\varphi] \right) \right\} d\varphi$$

$$= \frac{\exp\left(-\tfrac{1}{4} \frac{N \overline{a^2}}{\sigma^2} \right)}{2\pi} \cdot$$

$$\int_0^{2\pi} \exp\left\{ \frac{1}{\sigma^2} \sum_{n=0}^{N-1} x[n]\, a_n \cos\left(n\,\omega_{\mathrm{c}} + \varphi - \tfrac{1}{2}\pi \right) \right\} d\varphi$$

$$= \frac{\exp\left(-\tfrac{1}{4} \frac{N \overline{a^2}}{\sigma^2} \right)}{2\pi}$$

$$\cdot \int_0^{2\pi} \exp\left(\frac{1}{\sigma^2} \operatorname{Re}\left\{ \sum_{n=0}^{N-1} x[n]\, a_n e^{j\,(n\,\omega_{\mathrm{c}} + \varphi - \tfrac{1}{2}\pi)} \right\} \right) d\varphi$$

$$= \frac{\exp\left(-\tfrac{1}{4} \frac{N \overline{a^2}}{\sigma^2} \right)}{2\pi}$$

$$\cdot \int_{\text{any } 2\pi \text{ interval}} \exp\left(\frac{1}{\sigma^2} \operatorname{Re}\left\{ z(\boldsymbol{x}) \exp\left[j\,(\varphi - \tfrac{1}{2}\pi) \right] \right\} \right) d\varphi \quad (2)$$

where

$$z(\boldsymbol{x}) = \sum_{n=0}^{N-1} x[n] \, a_n \exp(j \, n \, \omega_{\mathrm{c}}).$$

Clearly, (2) does not depend on $\angle z(\boldsymbol{x})$ and is, therefore, a function of $z(\boldsymbol{x})$ only through its magnitude $|z(\boldsymbol{x})|$. Furthermore, (2) is an increasing function of $|z(\boldsymbol{x})|$, implying that we can simplify our test to

$$\left| \sum_{n=0}^{N-1} x[n] \, a_n \exp(j \, n \, \omega_{\mathrm{c}}) \right| \overset{\mathcal{H}_1}{\underset{}{\gtrless}} \text{ a threshold } \gamma \quad \Longleftrightarrow$$

$$\left| \underbrace{\sum_{n=0}^{N-1} x[n] \, a_n \exp(-j \, n \, \omega_{\mathrm{c}})}_{\text{Fourier transform of } x[n] \, a_n} \right| \overset{\mathcal{H}_1}{\underset{}{\gtrless}} \text{ a threshold} \quad \Longleftrightarrow$$

$$\left\{ \underbrace{\sum_{n=0}^{N-1} x[n] \, a_n \cos(n \, \omega_{\mathrm{c}})}_{\text{quadrature component}} \right\}^2 + \left\{ \underbrace{\sum_{n=0}^{N-1} x[n] \, a_n \sin(n \, \omega_{\mathrm{c}})}_{\text{quadrature component}} \right\}^2$$

$$\overset{\mathcal{H}_1}{\underset{}{\gtrless}} \text{ a threshold}$$

in both the Bayesian and Neyman-Pearson scenarios (as usual, only the choice of the threshold differs between the two scenarios). In this case, we can evaluate the integral (2)

exactly, yielding

$$\Lambda(\boldsymbol{x}) \approx \exp\Big(-\tfrac{1}{4}\frac{N\,\overline{a^2}}{\sigma^2} \Big) \cdot I_0\Big(\frac{|z(\boldsymbol{x})|}{\sigma^2} \Big)$$

where $I_0(\cdot)$ denotes the zeroth-order modified Bessel function of the first kind, which can be defined as follows:

$$I_0(|z|) = \frac{1}{2\pi} \cdot \int_0^{2\pi} e^{\mathrm{Re}\{ze^{j\varphi}\}}\, d\varphi.$$
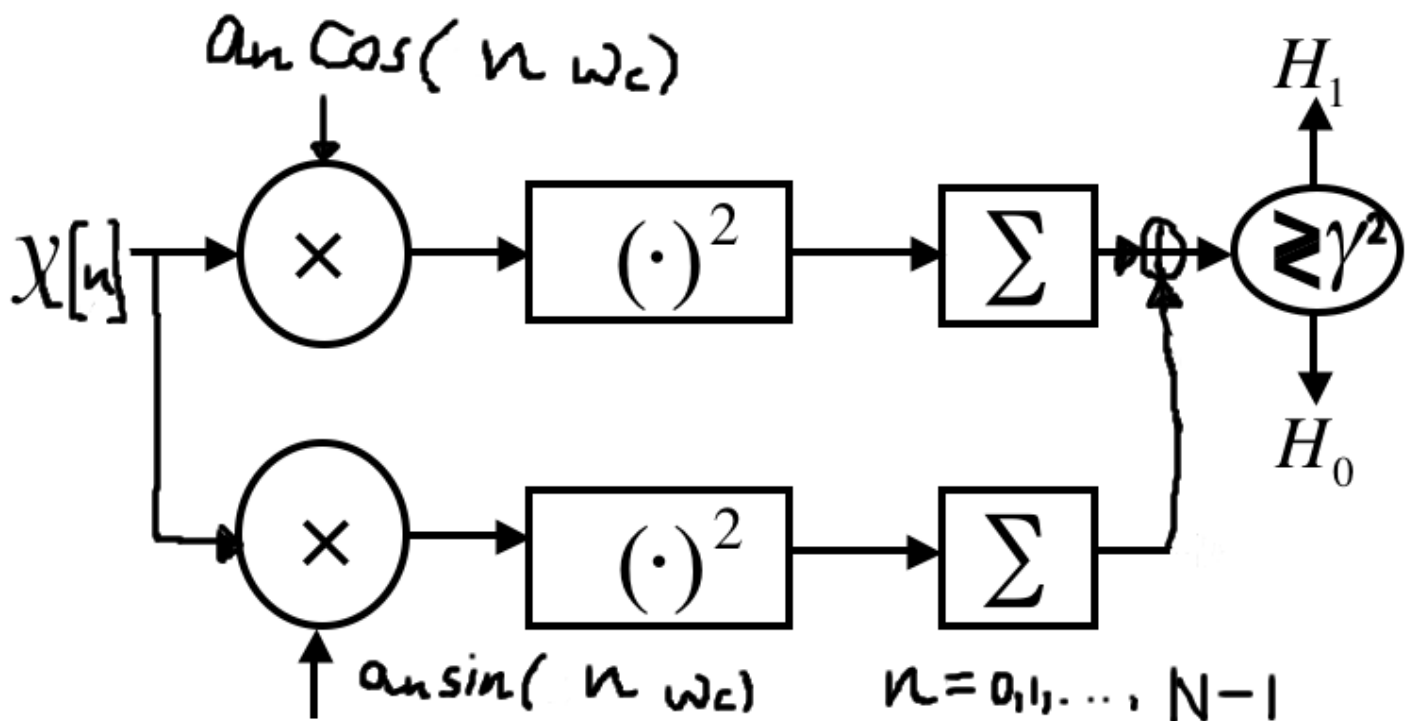
[Here, we can easily show that the right-hand side of the above expression is a function of $|z|$ only (i.e. independent of the phase $\angle z$):

$$\frac{1}{2\pi} \cdot \int_0^{2\pi} e^{\mathrm{Re}\{ze^{j\varphi}\}}\, d\varphi$$

$$= \quad \frac{1}{2\pi} \cdot \int_0^{2\pi} e^{\mathrm{Re}\{|z|\, e^{j(\angle z + \varphi)}\}}\, d\varphi$$

$$\overset{\theta = \angle z + \varphi}{=} \quad \underbrace{\frac{1}{2\pi} \cdot \int_{\text{any } 2\pi \text{ interval}} e^{\mathrm{Re}\{|z|\, e^{j\theta}\}}\, d\theta}_{\textcolor{red}{\text{a function of } |z| \text{ only}}}$$

$$= \quad \frac{1}{2\pi} \cdot \int_{\text{any } 2\pi \text{ interval}} e^{|z|\, \mathrm{Re}\{e^{j\theta}\}}\, d\theta$$

$$= \quad \int_0^{2\pi} e^{|z|\, \cos\theta}\, d\theta \overset{\triangle}{=} I_0(|z|).$$

] Therefore, our Bayes' decision rule simplifies to

$$|z(\boldsymbol{x})| \overset{\mathcal{H}_1}{\underset{}{\gtrless}} \underbrace{\sigma^2 \cdot I_0^{-1}\left(\exp\left(\frac{1}{4}\frac{N\,\overline{a^2}}{\sigma^2}\right) \cdot \frac{\pi_0\,\mathrm{L}(1\,|\,0)}{\pi_1\,\mathrm{L}(0\,|\,1)} \right)}_{\gamma}$$

leading to the following receiver structure:

Note: Here, to implement the the maximum-likelihood detection rule:

$$|z(\boldsymbol{x})| \overset{\mathcal{H}_1}{\underset{}{\gtrless}} \underbrace{\sigma^2 \cdot I_0^{-1}\left(\exp\left(\frac{1}{4}\frac{N\,\overline{a^2}}{\sigma^2}\right) \right)}_{\text{maximum-likelihood rule threshold}}$$

we need to know the AWGN noise variance σ^2.

Detecting A Stochastic Signal in AWGN (Neyman-Pearson Approach)

Some signals have unknown waveform (e.g. speech signals or NDE defect responses). We may need to use stochastic models to describe such signals. We start with a simple independent-signal model, described below.

Estimator-correlator. Consider the following hypothesis test:

$$\mathcal{H}_0: \quad x[n] = w[n], \quad n = 1, 2, \ldots, N$$
$$\mathcal{H}_1: \quad x[n] = s[n] + w[n], \quad n = 1, 2, \ldots, N$$

where

- $s[n]$ are zero-mean independent Gaussian random variables with *known* variances $\sigma_{s,n}^2$, i.e. $s[n] \sim \mathcal{N}(0, \sigma_{s,n}^2)$,

- the noise $w[n]$ is AWGN with *known variance* σ^2, i.e. $w[n] \sim \mathcal{N}(0, \sigma^2)$,

- $s[n]$ and $w[n]$ are independent.

Here is an alternative formulation. Consider the following

family of pdfs:

$$p(\boldsymbol{x} \,|\, C_s) = \mathcal{N}(\boldsymbol{0}, C_s + \sigma^2 \, I)$$

$$= \frac{1}{\sqrt{\prod_{n=1}^{N}[2\,\pi\,(\sigma^2 + c_{s,n})]}} \cdot \exp\left[-\sum_{n=1}^{N}\frac{(x[n])^2}{2\,(\sigma^2 + c_{s,n})}\right]$$

$$C_s \;=\; \text{diag}\{c_{s,1}, c_{s,2}, \ldots, c_{s,N}\}$$

with

$$\boldsymbol{x} = \begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ x[N] \end{bmatrix}$$

and the following (equivalent) hypotheses:

$$\mathcal{H}_0: \qquad C_s = 0 \quad \textcolor{red}{\text{(signal absent)}} \quad \text{versus}$$

$$\mathcal{H}_1: \qquad C_s = \text{diag}\{\sigma^2_{s,1}, \sigma^2_{s,2}, \ldots, \sigma^2_{s,N}\} \quad \textcolor{red}{\text{(signal present).}}$$

Clearly, we have *integrated $s[n]$, $n = 1, 2, \ldots, N$ out* to obtain the marginal likelihood $p(\boldsymbol{x} \,|\, \Sigma_s)$ under $\mathcal{H}_1$.

Here, the only discrimination between the two hypotheses is in variance of the measurements (i.e. power of the received signal). The Neyman-Pearson detector computes the likelihood ratio:

$$\Lambda(\boldsymbol{x}) = \frac{p(\boldsymbol{x} \,|\, \text{diag}\{\sigma^2_{s,1}, \sigma^2_{s,2}, \ldots, \sigma^2_{s,N}\})}{p(\boldsymbol{x} \,|\, 0)}.$$

Note that

$$\frac{1}{\sigma^2} - \frac{1}{\sigma^2 + \sigma_{s,n}^2} = \frac{1}{\sigma^2} \cdot \underbrace{\frac{\sigma_{s,n}^2}{\sigma^2 + \sigma_{s,n}^2}}_{\overset{\triangle}{=} \ \kappa_n}$$

where we define

$$\kappa_n \ \overset{\triangle}{=} \ \frac{\sigma_{s,n}^2}{\sigma^2 + \sigma_{s,n}^2}.$$

Let us compute the log likelihood ratio:

$$\log \Lambda(\boldsymbol{x}) \ = \ \underbrace{\text{const}}_{\color{red}{\text{not a function of } \boldsymbol{x}}}$$

$$- \sum_{n=1}^{N} \frac{(x[n])^2}{2\left(\sigma^2 + \sigma_{s,n}^2\right)} + \sum_{n=1}^{N} \frac{(x[n])^2}{2\,\sigma^2}$$

$$= \ \underbrace{\text{const}}_{\color{red}{\text{not a function of } \boldsymbol{x}}} + \frac{2}{\sigma^2} \cdot \sum_{n=1}^{N} \kappa_n \cdot (x[n])^2$$

and, therefore, our test simplifies to (after ignoring the constant terms and scaling the log likelihood ratio by the positive constant $\sigma^2/2$):
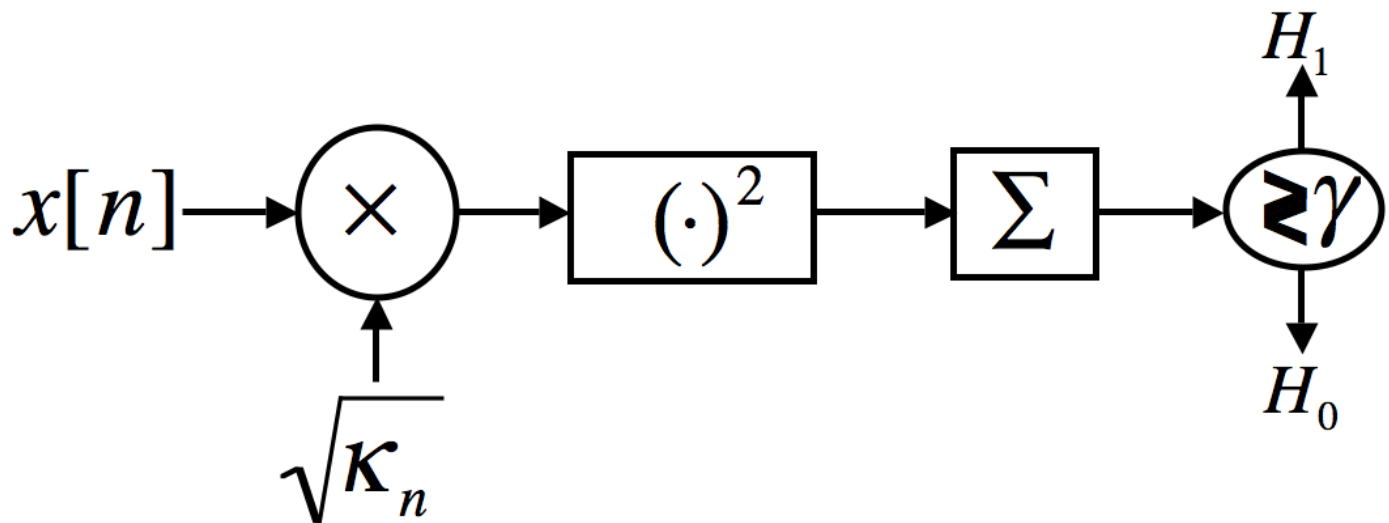
$$T(\boldsymbol{x}) = \sum_{n=1}^{N} \kappa_n \cdot (x[n])^2 \ \overset{\mathcal{H}_1}{\underset{}{\gtrless}} \ \gamma \quad \color{teal}{\text{(a threshold).}}$$

We first provide two interpretations of this detector and then

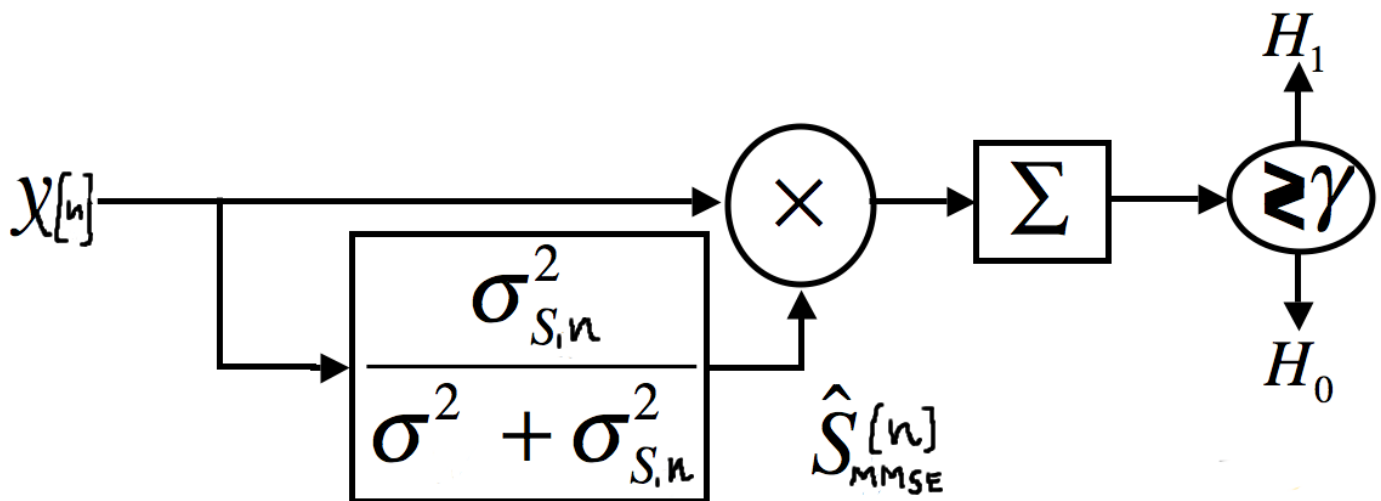generalize it to the case of correlated signal $s[n]$ having known covariance.

Filter-squarer interpretation:

$$\sum_{n=1}^{N} \kappa_n \cdot (x[n])^2 = \sum_{n=1}^{N} (\sqrt{\kappa_n} \cdot x[n])^2.$$

Estimator-correlator interpretation:

$$
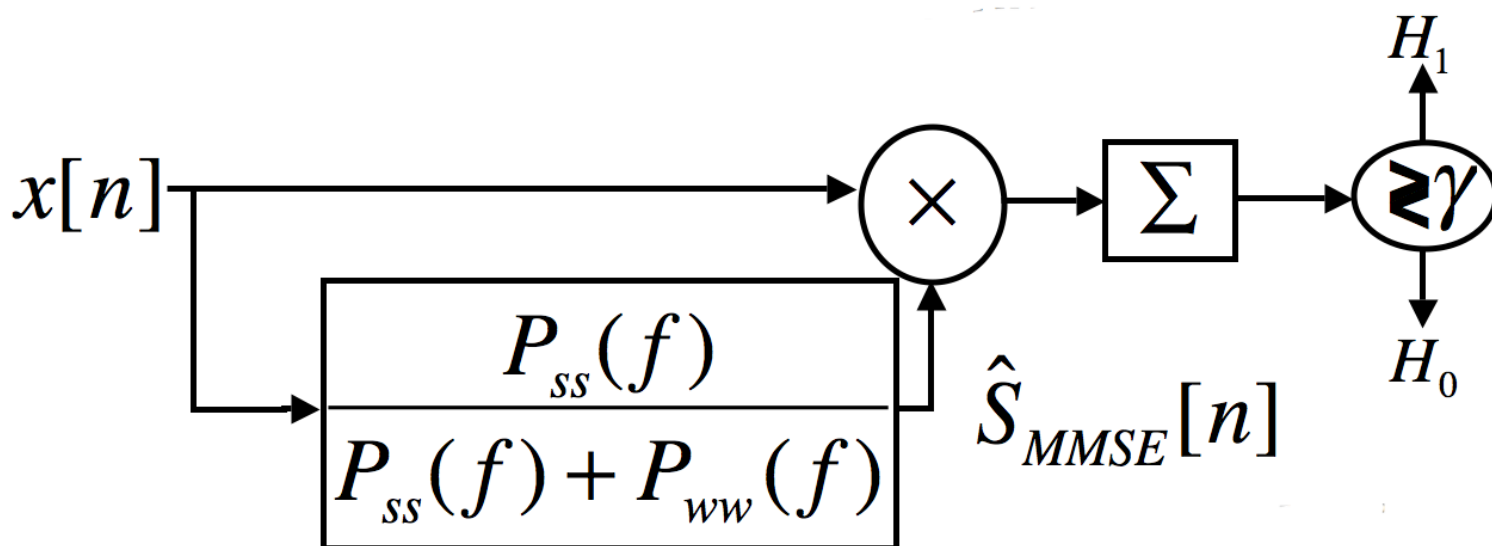\sum_{n=1}^{N} \kappa_n \cdot (x[n])^2 \;=\; \sum_{n=1}^{N} x[n] \cdot (\kappa_n \, x[n])
$$

$$
\;=\; \sum_{n=1}^{N} x[n] \cdot \underbrace{\frac{\sigma_{s,n}^2}{\sigma^2 + \sigma_{s,n}^2} \, x[n]}_{\mathrm{E}\,[s[n]\,|\,y[n]] = \widehat{s}_{\mathrm{MMSE}}[n]} \;.
$$

(Asymptotic) Estimator-correlator: Wide-sense Stationary (WSS) signal $s[n]$ in WSS noise $w[n]$. Suppose that $s[n]$ and $w[n]$ are zero-mean WSS sequences with power spectral densities (PSDs)

$$P_{ss}(f) \quad \text{and} \quad P_{ww}(f), \quad f \in [-\tfrac{1}{2}, \tfrac{1}{2}].$$

Then, our estimator-correlator structure remains the same, with the estimator block modified accordingly:

which is an asymptotic estimator-correlator.

Estimator-correlator for Correlated Signal in the Form of a Linear Model

We extend the estimator-correlator to the case of correlated signal:

$$\mathcal{H}_0: \qquad \boldsymbol{x} = \boldsymbol{w} \quad \text{versus}$$

$$\mathcal{H}_1: \qquad \boldsymbol{x} = \underbrace{H\boldsymbol{\theta}}_{\text{signal } \boldsymbol{s}} + \boldsymbol{w}$$

where

- H is a known $N \times p$ matrix and $N \geq p$.

- the noise $\boldsymbol{w}$ is zero-mean Gaussian with known covariance matrix $\Sigma_{\boldsymbol{w}}$:
$$p(\boldsymbol{w}) = \mathcal{N}(\boldsymbol{0}, \Sigma_{\boldsymbol{w}}).$$

- $\boldsymbol{\theta}$ is *unknown*, with the following prior pdf:

$$\pi(\boldsymbol{\theta}) = \mathcal{N}(\boldsymbol{0}, \Sigma_{\boldsymbol{\theta}})$$

where $\Sigma_{\boldsymbol{\theta}}$ is known.

We focus on the following general family of pdfs:

$$p(\boldsymbol{x} \,|\, C_s) = \mathcal{N}(\boldsymbol{0}, C_s + \Sigma_{\boldsymbol{w}})$$

where C_s is a positive-definite covariance matrix. Then, the above hypothesis-testing problem can be equivalently stated as

$$\mathcal{H}_0 : \qquad C_s = 0 \quad \text{(signal absent)} \quad \text{versus}$$
$$\mathcal{H}_1 : \qquad C_s = H\Sigma_{\boldsymbol{\theta}}H^T \quad \text{(signal present)}.$$

The estimator-correlator test statistic is now

$$T(\boldsymbol{x}) \;\; = \;\; \tfrac{1}{2}\boldsymbol{x}^T\,\Sigma_{\boldsymbol{w}}^{-1}\,\boldsymbol{x} - \tfrac{1}{2}\boldsymbol{x}^T\,(H\Sigma_{\boldsymbol{\theta}}H^T + \Sigma_{\boldsymbol{w}})^{-1}\,\boldsymbol{x}$$

Recall the matrix inversion lemma:

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}$$

and use it as follows:

$$(\Sigma_{\boldsymbol{w}} + H\Sigma_{\boldsymbol{\theta}}H^T)^{-1} = \Sigma_{\boldsymbol{w}}^{-1}$$
$$- \Sigma_{\boldsymbol{w}}^{-1}H(\Sigma_{\boldsymbol{\theta}}^{-1} + H^T\Sigma_{\boldsymbol{w}}^{-1}H)^{-1}H^T\Sigma_{\boldsymbol{w}}^{-1}$$

yielding

$$T(\boldsymbol{x}) = \tfrac{1}{2}\cdot\boldsymbol{x}^T\Sigma_{\boldsymbol{w}}^{-1}H\;\underbrace{(\Sigma_{\boldsymbol{\theta}}^{-1} + H^T\Sigma_{\boldsymbol{w}}^{-1}H)^{-1}H^T\Sigma_{\boldsymbol{w}}^{-1}\boldsymbol{x}}_{\mathrm{E}\,[\boldsymbol{\theta}\,|\,\boldsymbol{x}]\,\triangleq\,\widehat{\boldsymbol{\theta}}_{\mathrm{MMSE},}\ \text{see handout}\ \#\,4}\;.\quad(3)$$

Example: Detecting a Sinusoid in a Rayleigh-fading Channel (Neyman-Pearson Approach)

Over a short time interval, the channel output is a constant-amplitude sinusoid with random amplitude and phase, i.e.

$$s[n] = A\cos(2\pi f_0 n + \varphi) = a\,\cos(2\pi f_0 n) + b\,\sin(2\pi f_0 n)$$

for $n = 0, 1, \ldots, N-1$. Let us choose independent, identically distributed (i.i.d.) Rayleigh fading:

$$\boldsymbol{\theta} = \left[\begin{array}{c} a \\ b \end{array}\right] \sim \mathcal{N}(\mathbf{0}, \underbrace{\sigma_{\boldsymbol{\theta}}^2 I}_{\Sigma_{\boldsymbol{\theta}}})$$

implying

$$\begin{aligned} A &= \sqrt{a^2 + b^2} \sim \text{ a Rayleigh random variable} \\ \varphi &\sim \text{ uniform}(0, 2\pi). \end{aligned}$$

With these assumptions, $s[n]$ is a WSS Gaussian random

process, since

$$\mathrm{E}\left(s[n]\right) = 0$$

$$\mathrm{E}\left(s[n]s[n+k]\right) = \mathrm{E}\left\{[a\,\cos 2\pi f_0 n + b\,\sin 2\pi f_0 n]\right.$$

$$\cdot [a\,\cos 2\pi f_0 (n+k) + b\,\sin 2\pi f_0 (n+k)]\}$$

$$= \sigma_{\boldsymbol{\theta}}^2 \cdot [\cos 2\pi f_0 n \cos 2\pi f_0 (n+k)$$

$$+ \sin 2\pi f_0 n \sin 2\pi f_0 (n+k)]$$

$$= \sigma_{\boldsymbol{\theta}}^2 \cdot \left\{ \frac{\exp(j2\pi f_0 n) + \exp(-j2\pi f_0 n)}{2} \right.$$

$$\cdot \frac{\exp(j2\pi f_0 (n+k)) + \exp(-j2\pi f_0 (n+k))}{2}$$

$$+ \frac{\exp(j2\pi f_0 n) - \exp(-j2\pi f_0 n)}{2j}$$

$$\left. \cdot \frac{\exp(j2\pi f_0 (n+k)) - \exp(-j2\pi f_0 (n+k))}{2j} \right\}$$

$$= \sigma_{\boldsymbol{\theta}}^2 \cos 2\pi f_0 k = r_{ss}[k]$$

for $n = 0, 1, \ldots, N-1$. We now construct a linear model with

$$H = \begin{bmatrix} 1 & 0 \\ \cos 2\pi f_0 & \sin 2\pi f_0 \\ \vdots & \vdots \\ \cos 2\pi f_0 (N-1) & \sin 2\pi f_0 (N-1) \end{bmatrix}$$

and
$$\Sigma_{\boldsymbol{w}} = \sigma^2 I \quad \text{(AWGN)}.$$

Note that σ^2 and $\sigma_{\boldsymbol{\theta}}^2$ are assumed *known*. Now, (3) reduces to

$$
\begin{aligned}
T(\boldsymbol{x}) &= \tfrac{1}{2} \cdot \boldsymbol{x}^T \Sigma_{\boldsymbol{w}}^{-1} H (\Sigma_{\boldsymbol{\theta}}^{-1} + H^T \Sigma_{\boldsymbol{w}}^{-1} H)^{-1} H^T \Sigma_{\boldsymbol{w}}^{-1} \boldsymbol{x} \\
&= \frac{1}{2\,(\sigma^2)^2} \cdot \boldsymbol{x}^T H\,(\sigma_{\boldsymbol{\theta}}^{-2} I + \sigma^{-2} H^T H)^{-1} H^T \boldsymbol{x}
\end{aligned}
$$

For large N and f_0 not too close to 0 or $\tfrac{1}{2}$,

$$H^T H \approx (N/2)\,I$$

see p. 157 in Kay-II. Furthermore,

$$
H^T \boldsymbol{x} = \begin{bmatrix} \sum_{n=0}^{N-1} x[n] \cos 2\pi f_0 n \\ \sum_{n=0}^{N-1} x[n] \sin 2\pi f_0 n \end{bmatrix}.
$$

After scaling by the positive constant $2\,(\sigma^2)^2/(\sigma_{\boldsymbol{\theta}}^{-2} + \sigma^{-2}\,N/2)$, our test statistic $T(\boldsymbol{x})$ simplifies to

$$
T'(\boldsymbol{x}) = \frac{1}{N} \left| \sum_{n=0}^{N-1} x[n] \exp(-j2\pi f_0 n) \right|^2
$$

which is nothing but the periodogram of $x[n]$, $n = 1, 2, \ldots, N$ evaluated at frequency $f = f_0$, also known as the *quadrature or noncoherent matched filter*.

Performance Analysis for the Neyman-Pearson Setup:
Define

$$\boldsymbol{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = H^T \boldsymbol{x}.$$

Under $\mathcal{H}_0$, we have only noise, implying that

$$\boldsymbol{\xi} \sim \mathcal{N}(\boldsymbol{0}, H^T \, \sigma^2 \, I \, H \approx \underbrace{\frac{N\sigma^2}{2}}_{\triangleq \, s_0^2} I).$$

Under $\mathcal{H}_1$, we have

$$\boldsymbol{\xi} \sim \mathcal{N}\Big(\boldsymbol{0}, H^T(H \underbrace{\Sigma_{\boldsymbol{\theta}}}_{\sigma_{\boldsymbol{\theta}}^2 I} H^T + \sigma^2 I)H\Big).$$

We approximate the covariance matrix of $\boldsymbol{\xi}$ under $\mathcal{H}_1$ as follows:

$$
\begin{aligned}
H^T(H \Sigma_{\boldsymbol{\theta}} H^T + \sigma^2 I)H \quad &\approx \quad \sigma_{\boldsymbol{\theta}}^2 \, (N/2)^2 \, I + \sigma^2 \, (N/2) \, I \\
&= \quad \underbrace{\frac{N}{2}\left(\frac{N}{2}\sigma_{\boldsymbol{\theta}}^2 + \sigma^2\right)}_{\triangleq \, s_1^2} I.
\end{aligned}
$$

Now, under $\mathcal{H}_0$,

$$T'(\boldsymbol{x}) = \frac{1}{N}\,(\xi_1^2 + \xi_2^2) = \frac{s_0^2}{N}\,\underbrace{\left[\left(\overbrace{\frac{\xi_1}{s_0}}^{\text{standard normal random variable}}\right)^2 + \left(\frac{\xi_2}{s_0}\right)^2\right]}_{\chi_2^2 \quad \text{under } \mathcal{H}_0}$$

implying that

$$\begin{aligned}
P_{\text{FA}} &= P[T'(\boldsymbol{X}) > \gamma \,|\, C_s = 0] \\[2mm]
&= P\Big[\overbrace{\underbrace{\frac{N\,T'(\boldsymbol{X})}{s_0^2}}_{\chi_2^2} > \frac{N\,\gamma}{s_0^2}}^{Q_{\chi_2^2}\left(\frac{N\,\gamma}{s_0^2}\right)}\,\Big|\, C_s = 0\Big] \\[2mm]
&= \exp(-\tfrac{1}{2}\,N\,\gamma/s_0^2)
\end{aligned}$$

see eq. (2.10) in Kay-II. Similarly, under $\mathcal{H}_1$,

$$\begin{aligned}
P_{\text{D}} &= P[T'(\boldsymbol{X}) > \gamma \,|\, C_s = \sigma_{\boldsymbol{\theta}}^2\,HH^T] \\[2mm]
&= P\Big[\frac{N T'(\boldsymbol{X})}{s_1^2} > \frac{N\gamma}{s_1^2}\,\Big|\, C_s = \sigma_{\boldsymbol{\theta}}^2\,HH^T\Big] \\[2mm]
&= \exp(-\tfrac{1}{2}\,N\,\gamma/s_1^2).
\end{aligned}$$

But

$$-\tfrac{1}{2}\,N\,\gamma = s_0^2 \log P_{\mathrm{FA}}$$

leading to

$$P_{\mathrm{D}} = \exp\left(\frac{s_0^2}{s_1^2} \log P_{\mathrm{FA}}\right) = P_{\mathrm{FA}}{}^{s_0^2/s_1^2}.$$

Recall the expressions for s_0^2 and s_1^2 and compute their ratio:

$$\frac{s_1^2}{s_0^2} = \frac{\frac{N}{2}\left(\frac{N}{2}\sigma_{\boldsymbol\theta}^2 + \sigma^2\right)}{\frac{N\sigma^2}{2}} = \frac{N}{2}\frac{\sigma_{\boldsymbol\theta}^2}{\sigma^2} + 1 = \frac{\overline{\eta}}{2} + 1$$

where

$$\overline{\eta} = \frac{N\sigma_{\boldsymbol\theta}^2}{\sigma^2} = \frac{N\mathrm{E}[\,\overbrace{A^2}^{a^2+b^2}\,/2]}{\sigma^2} \equiv \text{average SNR.}$$

Hence

$$P_{\mathrm{D}} = P_{\mathrm{FA}}{}^{\frac{1}{1+\overline{\eta}/2}}.$$

P_{D} increases slowly with the average signal-to-noise ratio (SNR) $\overline{\eta}$ (see Figure 5.7 in Kay-II) because Rayleigh fading causes amplitude to be small with high probability.

Coherent channel $\Rightarrow$ matched filter.

Noncoherent channel $\Rightarrow$ quadrature matched filter.

Compare Figures 5.7 and 4.5 in Kay-II.

Noncoherent FSK in a Rayleigh-fading Channel (Bayesian decision-theoretic detection for 0-1 loss)

$$
\mathcal{H}_0 : x[n] = A \cos(2\pi f_0 n + \varphi) + w[n], \quad n = 0, 1, \ldots, N - 1
$$
$$
\mathcal{H}_1 : x[n] = A \cos(2\pi f_1 n + \varphi) + w[n], \quad n = 0, 1, \ldots, N - 1
$$

where, as before, A and φ are random phase and amplitude and

$$
\boldsymbol{w} = \begin{bmatrix} w[0] \\ w[1] \\ \vdots \\ w[N-1] \end{bmatrix} \sim \mathcal{N}(\mathbf{0}, \Sigma_{\boldsymbol{w}}).
$$

We now focus on the following family of pdfs:

$$
p(\boldsymbol{x} \,|\, H) = \mathcal{N}(\mathbf{0}, H \,\Sigma_{\boldsymbol{\theta}}\, H^T + \Sigma_{\boldsymbol{w}})
$$

where $\Sigma_{\boldsymbol{\theta}}$ is known. We can rewrite the above detection problem using the linear model as follows:

$$
\mathcal{H}_0 : \qquad H = H_0 \quad \text{versus}
$$
$$
\mathcal{H}_1 : \qquad H = H_1
$$

where $\boldsymbol{x} = \begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ x[N] \end{bmatrix}$,

$$H_0 = \begin{bmatrix} 1 & 0 \\ \cos 2\pi f_0 & \sin 2\pi f_0 \\ \vdots & \vdots \\ \cos 2\pi f_0(N-1) & \sin 2\pi f_0(N-1) \end{bmatrix}$$

and

$$H_1 = \begin{bmatrix} 1 & 0 \\ \cos 2\pi f_1 & \sin 2\pi f_1 \\ \vdots & \vdots \\ \cos 2\pi f_1(N-1) & \sin 2\pi f_1(N-1) \end{bmatrix}.$$

For *a priori* equiprobable hypotheses

$$\pi(H = H_0) = \pi(H = H_1) = \tfrac{1}{2}$$

we have the maximum-likelihood test:

$$\frac{p(\boldsymbol{x} \mid H = H_1)}{p(\boldsymbol{x} \mid H = H_0)} \overset{\mathcal{H}_1}{\underset{}{\gtrless}} 1$$

i.e.

$$\frac{\frac{1}{|H_1\Sigma_{\boldsymbol{\theta}}H_1^T+\Sigma_{\boldsymbol{w}}|^{1/2}} \exp\left[\frac{1}{2}\boldsymbol{x}^T\Sigma_{\boldsymbol{w}}^{-1}\boldsymbol{x} - \frac{1}{2}\boldsymbol{x}^T(H_1\Sigma_{\boldsymbol{\theta}}H_1^T+\Sigma_{\boldsymbol{w}})^{-1}\boldsymbol{x}\right]}{\frac{1}{|H_0\Sigma_{\boldsymbol{\theta}}H_0^T+\Sigma_{\boldsymbol{w}}|^{1/2}} \exp\left[\frac{1}{2}\boldsymbol{x}^T\Sigma_{\boldsymbol{w}}^{-1}\boldsymbol{x} - \frac{1}{2}\boldsymbol{x}^T(H_0\Sigma_{\boldsymbol{\theta}}H_0^T+\Sigma_{\boldsymbol{w}})^{-1}\boldsymbol{x}\right]} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} 1$$

which can be written as

$$\frac{\frac{1}{|H_1\Sigma_{\boldsymbol{\theta}}H_1^T+\Sigma_{\boldsymbol{w}}|^{1/2}} \exp\left[\frac{1}{2}\cdot\boldsymbol{x}^T\Sigma_{\boldsymbol{w}}^{-1}H_1(\Sigma_{\boldsymbol{\theta}}^{-1}+H_1^T\Sigma_{\boldsymbol{w}}^{-1}H_1)^{-1}H_1^T\Sigma_{\boldsymbol{w}}^{-1}\boldsymbol{x}\right]}{\frac{1}{|H_0\Sigma_{\boldsymbol{\theta}}H_0^T+\Sigma_{\boldsymbol{w}}|^{1/2}} \exp\left[\frac{1}{2}\cdot\boldsymbol{x}^T\Sigma_{\boldsymbol{w}}^{-1}H_0(\Sigma_{\boldsymbol{\theta}}^{-1}+H_0^T\Sigma_{\boldsymbol{w}}^{-1}H_0)^{-1}H_0^T\Sigma_{\boldsymbol{w}}^{-1}\boldsymbol{x}\right]} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} 1.$$

To further simplify the above test, we adopt additional assumptions. In particular, we assume i.i.d. Rayleigh fading:

$$\Sigma_{\boldsymbol{\theta}} = \sigma_{\boldsymbol{\theta}}^2\,I$$

and AWGN

$$\Sigma_{\boldsymbol{w}} = \sigma^2\,I \quad \text{(AWGN)}.$$

where σ^2 and $\sigma_{\boldsymbol{\theta}}^2$ are *known*. For large N and f_i, $i \in \{0,1\}$ not too close to 0 or $\frac{1}{2}$,

$$H_i^T H_i \approx (N/2)\,I$$

see p. 157 in Kay-II. Now, we apply this approximation and the identities $|PQ| = |P|\cdot|Q|$ and $|I+AB| = |I+BA|$ to further

simplify the above determinant expressions:

$$
\begin{aligned}
|H_i \Sigma_{\boldsymbol{\theta}} H_i^T + \Sigma_{\boldsymbol{w}}| &= |\sigma_{\boldsymbol{\theta}}^2 H_0 H_0^T + \sigma^2 I| \\
&= |\sigma^2 I| \cdot |I + \frac{\sigma_{\boldsymbol{\theta}}^2}{\sigma^2} H_0 H_0^T| \\
&\approx (\sigma^2)^N \cdot \left| I_2 + \frac{\sigma_s^2}{\sigma^2} \frac{N}{2} I_2 \right|
\end{aligned}
$$

for $i \in \{0, 1\}$. Applying the above approximation and assumptions yields the simplified maximum-likelihood test:

$$
\frac{\exp\left[\frac{1}{2\sigma^4} \cdot \boldsymbol{x}^T H_1 \left(\sigma_{\boldsymbol{\theta}}^2 I + \frac{N}{2\sigma^2} I\right)^{-1} H_1^T \boldsymbol{x}\right]}{\exp\left[\frac{1}{2\sigma^4} \cdot \boldsymbol{x}^T H_0 \left(\sigma_{\boldsymbol{\theta}}^2 I + \frac{N}{2\sigma^2} I\right)^{-1} H_0^T \boldsymbol{x}\right]} \overset{\mathcal{H}_1}{\underset{}{\gtrless}} 1.
$$

and, equivalently,

$$
\boldsymbol{x}^T H_1 H_1^T \boldsymbol{x} - \boldsymbol{x}^T H_0 H_0^T \boldsymbol{x} \overset{\mathcal{H}_1}{\gtrless} 0
$$

or

$$
\frac{1}{N} \left| \sum_{n=0}^{N-1} x[n] \exp(-j2\pi f_1 n) \right|^2 \overset{\mathcal{H}_1}{\gtrless} \frac{1}{N} \left| \sum_{n=0}^{N-1} x[n] \exp(-j2\pi f_0 n) \right|^2.
$$

For performance analysis of this detector (i.e. computing an approximate expression for the average error probability), see Example 5.6 in Kay-II.

Preliminaries

Let us define a matrix-variate circularly symmetric complex Gaussian pdf of an $p \times q$ random matrix Z with mean M (of size $p \times q$) and positive-definite covariance matrices S and Σ (of dimensions $p \times p$ and $q \times q$, respectively) as follows:

$$
\begin{aligned}
\mathcal{N}_{p \times q}(Z \,|\, M, S, \Sigma) &= \frac{1}{\pi^{pq} \, |S|^q \, |\Sigma|^p} \\
&\quad \cdot \exp \big\{ -\operatorname{tr}[\Sigma^{-1}(Z-M)^H S^{-1}(Z-M)] \big\} \\
&\propto \exp \big\{ -\operatorname{tr}[\Sigma^{-1}(Z-M)^H S^{-1}(Z-M)] \big\} \\
&\propto \exp \big[-\operatorname{tr}(\Sigma^{-1} Z^H S^{-1} Z) + \operatorname{tr}(\Sigma^{-1} Z^H S^{-1} M) \\
&\quad + \operatorname{tr}(\Sigma^{-1} M^H S^{-1} Z) \big]
\end{aligned}
$$

where "H" denotes the Hermitian (conjugate) transpose.

Noncoherent Detection of Space-time Codes in a Rayleigh Fading Channel

Consider the multiple-input multiple-output (MIMO) flat-fading channel where the $n_{\mathrm{R}} \times 1$ vector signal received by the receiver array at time t is modeled as

$$\boldsymbol{x}(t) = H\boldsymbol{\phi}(t) + \boldsymbol{w}(t), \quad t = 1, \ldots, N$$

where H is the $n_{\mathrm{R}} \times n_{\mathrm{T}}$ channel-response matrix, $\boldsymbol{\phi}(t)$ is the $n_{\mathrm{T}} \times 1$ vector of symbols transmitted by n_{T} transmitter antennas and received by the receiver array at time t, and $\boldsymbol{w}(t)$ is additive noise. Note that we can write the above model as

$$\underbrace{[\boldsymbol{x}(1) \cdots \boldsymbol{x}(N)]}_{X} = H \underbrace{[\boldsymbol{\phi}(1) \cdots \boldsymbol{\phi}(N)]}_{\Phi} + \underbrace{[\boldsymbol{w}(1) \cdots \boldsymbol{w}(N)]}_{W} \quad (4)$$

Here, Φ is a space-time code (multivariate "symbol") belonging to an M-ary constellation:

$$\Phi \in \{\Phi_0, \Phi_1, \ldots, \Phi_{M-1}\}.$$

We adopt *a priori* equiprobable hypotheses:[1]

$$\pi(\Phi) = \frac{1}{M} \cdot i_{\{\Phi_0, \Phi_1, \ldots, \Phi_{M-1}\}}(\Phi)$$

and assume that $\boldsymbol{w}(t)$, $t = 1, 2, \ldots, N$ are zero-mean i.i.d. with known covariance

$$\mathrm{cov}(\boldsymbol{w}(t)) = \sigma^2 \, I_{n_\mathrm{R}}.$$

Now, the likelihood function for the measurement model (4) is

$$p(X \mid \Phi, H) = \mathcal{N}_{n_\mathrm{R} \times N}(X \mid H\,\Phi, \sigma^2\, I, I)$$

$$= \frac{1}{\pi^{n_\mathrm{R} N} \, |\sigma^2 I|^N} \cdot \exp\left\{ -\frac{1}{\sigma^2}\, \mathrm{tr}[(X - H\,\Phi)^H (X - H\,\Phi)] \right\}.$$

We assume that Φ and H are independent *a priori*, i.e.

$$\pi(\Phi, H) = \pi(\Phi)\, \pi(H)$$

where $\pi(\Phi)$ is given in (5) and $\pi(H)$ is chosen according to

---

[1]Here, $i_A(x)$ denotes the indicator function:

$$i_A(x) = \left\{ \begin{array}{ll} 1, & x \in A, \\ 0, & \text{otherwise} \end{array} \right. .$$

the following separable Rayleigh-fading model:

$$\pi(H) \;=\; \mathcal{N}_{n_{\mathrm{R}} \times n_{\mathrm{T}}}(H \,|\, 0, I, \overbrace{\Delta_h}^{\text{transmitter fading corr. matrix}})$$

$$=\; \frac{1}{\pi^{n_{\mathrm{R}} n_{\mathrm{T}}} \, |\Delta_h|^{n_{\mathrm{R}}}} \cdot \exp\big[-\operatorname{tr}(\Delta_h^{-1} H^H H)\big].$$

Now

$$p(\Phi, H \,|\, X) \propto p(X \,|\, \Phi, H)\, \pi(\Phi)\, \pi(H)$$

$$\propto \;\; \exp\Big[\frac{1}{\sigma^2}\operatorname{tr}(\Phi^H H^H X) + \frac{1}{\sigma^2}\operatorname{tr}(X^H H \Phi)$$

$$\qquad -\frac{1}{\sigma^2}\operatorname{tr}(\Phi^H H^H H \Phi)\Big] \cdot i_{\{\Phi_0, \Phi_1, \ldots, \Phi_{M-1}\}}(\Phi)$$

$$\qquad \cdot \exp\big[-\operatorname{tr}(\Delta_h^{-1} H^H H)\big]$$

$$\propto \;\; \exp\Big\{\frac{1}{\sigma^2}\operatorname{tr}(\Phi^H H^H X) + \frac{1}{\sigma^2}\operatorname{tr}(X^H H \Phi)$$

$$\qquad -\operatorname{tr}\Big[\underbrace{\Big(\frac{1}{\sigma^2}\,\Phi\,\Phi^H + \Delta_h^{-1}\Big)}_{C_H(\Phi)^{-1}} H^H H\Big]\Big\} \cdot i_{\{\Phi_0, \Phi_1, \ldots, \Phi_{M-1}\}}(\Phi)$$

implying that

$$p(H \mid \Phi, X)$$

$$\propto \quad \exp\left\{\frac{1}{\sigma^2}\,\mathrm{tr}(\Phi^H H^H X) + \frac{1}{\sigma^2}\,\mathrm{tr}(X^H H \Phi)\right.$$

$$\left. - \mathrm{tr}\left[\left(\frac{1}{\sigma^2}\,\Phi\,\Phi^H + \Delta_h^{-1}\right)H^H H\right]\right\}$$

$$\propto \quad \exp\left[-\mathrm{tr}\left\{C_H(\Phi)^{-1}\,H^H\,H\right\} + \mathrm{tr}\left\{C_H(\Phi)^{-1}\,\widehat{H}(\Phi)^H\,H\right\}\right.$$

$$\left. + \mathrm{tr}\left\{C_H(\Phi)^{-1}\,H^H\,\widehat{H}(\Phi)\right\}\right]$$

$$= \quad \mathcal{N}_{n_{\mathrm{R}} \times n_{\mathrm{T}}}\left(H \mid \underbrace{\frac{1}{\sigma^2}\,X\,\Phi^H\,\left(\frac{1}{\sigma^2}\,\Phi\,\Phi^H + \Delta_h^{-1}\right)^{-1}}_{\overset{\triangle}{=}\,\widehat{H}(\Phi)}, I,\right.$$

$$\underbrace{\left(\frac{1}{\sigma^2}\,\Phi\,\Phi^H + \Delta_h^{-1}\right)^{-1}}_{\overset{\triangle}{=}\,C_H(\Phi)}\right)$$

$$= \quad \frac{1}{\pi^{n_{\mathrm{R}} n_{\mathrm{T}}}\,|C_H(\Phi)|^{n_{\mathrm{R}}}}$$

$$\cdot \exp\left(-\mathrm{tr}\left\{C_H(\Phi)^{-1}\,[H - \widehat{H}(\Phi)]^H\,[H - \widehat{H}(\Phi)]\right\}\right)$$

where we have defined

$$
\begin{aligned}
\widehat{H}(\Phi) &\triangleq \frac{1}{\sigma^2} X \, \Phi^H \, C_H(\Phi) \\
C_H(\Phi) &\triangleq \left(\frac{1}{\sigma^2} \, \Phi \, \Phi^H + \Delta_h^{-1} \right)^{-1}
\end{aligned}
$$

Note the following useful facts:

$$
\begin{aligned}
\operatorname{tr}[C_H(\Phi)^{-1} H^H \widehat{H}(\Phi)] &= \frac{1}{\sigma^2} \operatorname{tr}[C_H(\Phi)^{-1} H^H X \, \Phi^H \, C_H(\Phi)] \\
&= \frac{1}{\sigma^2} \operatorname{tr}(H^H X \, \Phi^H) \\
\operatorname{tr}[C_H(\Phi)^{-1} \widehat{H}(\Phi)^H H] &= \frac{1}{\sigma^2} \operatorname{tr}[C_H(\Phi)^{-1} C_H(\Phi) \Phi X^H H] \\
&= \frac{1}{\sigma^2} \operatorname{tr}(\Phi X^H H).
\end{aligned}
$$

To obtain the marginal posterior pmf of Φ, we apply our

"notorious" trick:

$$p(\Phi \mid X) = \frac{p(\Phi, H \mid X)}{p(H \mid \Phi, X)}$$

$$\underbrace{\propto}$$

keep track of the terms containing Φ and H

$$\exp\left[\frac{1}{\sigma^2}\,\mathrm{tr}(\Phi^H H^H X) + \frac{1}{\sigma^2}\,\mathrm{tr}(X^H H \Phi) - \mathrm{tr}(C_H(\Phi)^{-1} H^H H)\right]$$

$$\cdot i_{\{\Phi_0, \Phi_1, \ldots, \Phi_{M-1}\}}(\Phi) \cdot |C_H(\Phi)|^{n_R}$$

$$\cdot \exp\left\{\mathrm{tr}\left[C_H(\Phi)^{-1}(H - \widehat{H}(\Phi))^H (H - \widehat{H}(\Phi))\right]\right\}$$

$$\propto\quad i_{\{\Phi_0, \Phi_1, \ldots, \Phi_{M-1}\}}(\Phi)$$

$$\cdot |C_H(\Phi)|^{n_R} \cdot \exp\{\mathrm{tr}[C_H(\Phi)^{-1}\widehat{H}(\Phi)^H \widehat{H}(\Phi)]\}$$

$$=\quad i_{\{\Phi_0, \Phi_1, \ldots, \Phi_{M-1}\}}(\Phi) \cdot |C_H(\Phi)|^{n_R}$$

$$\cdot \exp\left\{\frac{1}{(\sigma^2)^2}\,\mathrm{tr}[C_H(\Phi)^{-1}C_H(\Phi)\,\Phi\,X^H X\,\Phi^H\,C_H(\Phi)]\right\}$$

$$=\quad i_{\{\Phi_0, \Phi_1, \ldots, \Phi_{M-1}\}}(\Phi) \cdot |C_H(\Phi)|^{n_R}$$

$$\cdot \exp\left\{\frac{1}{(\sigma^2)^2}\,\mathrm{tr}[\Phi\,X^H X\,\Phi^H\,C_H(\Phi)]\right\}$$

and the maximum-likelihood test becomes:

$$
\mathcal{X}_m^\star = \Big\{ \boldsymbol{x} : m \;=\; \arg \max_{l \in \{0,1,\dots,M-1\}} \Big(|C_H(\varPhi_l)|^{n_{\mathrm{R}}}
$$

$$
\cdot \exp \Big(\frac{1}{(\sigma^2)^2} \; \mathrm{tr}[\varPhi_l \, X^H \, X \, \varPhi_l^H \, C_H(\varPhi_l)] \Big) \Big) \Big\}.
$$

Unitary space-time codes and i.i.d. fading: Suppose that $\varPhi_m \varPhi_m^H = I_{n_{\mathrm{T}}}$ for all m and the fading is i.i.d., i.e.

$$
\Delta_h = \psi_h^2 \, I_{n_{\mathrm{T}}}.
$$

Then, the above maximum-likelihood test greatly simplifies:

$$
\mathcal{X}_m^\star = \Big\{ \boldsymbol{x} : m \;=\; \arg \max_{l \in \{0,1,\dots,M-1\}} \mathrm{tr}[\varPhi_l \, X^H \, X \, \varPhi_l^H] \Big\}
$$

which is the detector proposed in

B.M. Hochwald and T.L. Marzetta, *IEEE Trans. Inform. Theory*, vol. 46, pp. 543–564, March 2000.