Homework 2: Deriving gradient flows using Calculus of Variations

1) Deriving Euler-Lagrange when E is a function of $u = u(x, y)$, i.e. u is a function of 2 variables (Application: optical flow).

   Given that

$$E(u) = \int_{y=a2}^{b2} \int_{x=a1}^{b1} L(u, u_x, u_y) dxdy \tag{1}$$

   Show that

$$(\nabla_u E) = (\frac{\partial L}{\partial u}) - \frac{\partial}{\partial x}\frac{\partial L}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial L}{\partial u_y} \tag{2}$$

   Assume $u(a1, y) = \gamma_1,\ \forall y$, $u(b1, y) = \gamma_2,\ \forall y$, $u(x, a2) = \gamma_3,\ \forall x$, $u(x, b2) = \gamma_4,\ \forall x$ (basically u is fixed at known values at the boundaries).

2) Deriving Euler Lagrange when E is a function of both first and second derivatives (Application: energy minimization for parametric snakes - Kass,Witkin,Terzopolous).

   Given that

$$E(C) = \int_{p=0}^{1} L(C, C_p, C_{pp}) dp \tag{3}$$

   Show that

$$(\nabla_C E) = (\frac{\partial L}{\partial C}) - \frac{\partial}{\partial p}\frac{\partial L}{\partial C_p} + \frac{\partial^2}{\partial p^2}\frac{\partial L}{\partial C_{pp}} \tag{4}$$

   Assume $C(0) = \gamma_1$, $C(1) = \gamma_2$, $C_p(0) = \gamma_3$, $C_p(1) = \gamma_4$ (basically C and C_p are fixed at known values at the boundaries).

   Note I have given the above assumption to simplify your homework problem. Actually for closed contours, do not need this assumption. First two terms in the "integration by parts" step cancel because $p = 0$ and $p = 1$ are the same point, so any function of p has the same value at both $p = 0$ and $p = 1$.

3) *Extra: (An exercise in understanding papers, both of these have been done in the papers posted on the class webpage)*

 a) Derive the edge based geometric active contour flow. Given

$$E(C) = \int_{s=0}^{L(C)} g(||\nabla I||)ds \tag{5}$$

show that

$$\nabla_C E = g(||\nabla I(C)||)\kappa N - (\nabla g \cdot N)N \tag{6}$$

where $L(C)$ denotes length of contour, N denotes the normal and κ denotes curvature.

Main idea: In the expression for E, the integral is over a region ($s = 0$ to $s = L(C)$) that depends on the contour itself. Hence cannot apply the formulae derived earlier. Solution: Assume an arbitrary parametrization $C(p)$ with $p \in [0, 1]$ and $C(0) = C(1)$. Then $ds = ||C_p||dp$ and the integral runs from $p = 0$ to $p = 1$. Then apply the standard formula

 b) Derive the region based geometric active contour flow. Given

$$E(C) = \int_{C_{inside}} (I(x, y) - u)^2 dxdy + \int_{C_{outside}} (I(x, y) - u)^2 dxdy + \alpha \int_{s=0}^{L(C)} ds \tag{7}$$

show that

$$\nabla_C E = (u - v)(I(C) - \frac{u + v}{2})N + \alpha\kappa N \tag{8}$$

NOTE: There may be a minus sign in the above, please verify

Main idea: The expression for E is a region integral depending on the contour. First convert it to a boundary integral over the contour boundary (using divergence theorem: $\int_R (\nabla \cdot F(x, y))dxdy = \int_{s=0}^{L(C)} (F(C) \cdot N)ds$ where C is the boundary of region R). Then use the idea described in part 3a.

Two things to note in the above:

When parameterizing contours by arc-length, s, the definition of "inner product" is: For 2 vectors $h_1(s)$ and $h_2(s)$ the inner product is $h_1 \cdot h_2 = \int_{s=0}^{L(C)} h_1(s)h_2(s)ds = \int_{p=0}^{1} h_1(p)h_2(p)||C_p||dp$.

Also note that there is abuse of notation when I also define the inner product in $\mathbb{R}^2$ by the same "$\cdot$" notation: for e.g. $\nabla g \cdot N$ is an inner product in $\mathbb{R}^2$.