# EE 424 #1: Sampling and Reconstruction

*January 13, 2011*

## Contents



READING: EE 224 handouts 2, 16, 18, 19, and `lctftsummary` (review); § 1.2.1, § 2.2.2, § 4.3, and § 7.1–§ 7.3 in the textbook[1].

[1] A. V. Oppenheim and A. S. Willsky. *Signals & Systems*. Prentice Hall, Upper Saddle River, NJ, 1997

## *Notation and Definitions*

**Definition 1.** *The unit rectangle is defined in Fig. 1.*

**Definition 2.** *The sinc function is defined as*

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \tag{1}$$

*see also Fig. 2.*

**Definition 3.** *An indicator function is defined as:*

$$\mathbb{1}_{(a,b)}(t) = \begin{cases} 1, & t \in (a,b) \\ 0, & \textit{otherwise} \end{cases}. \tag{2}$$

**Definition 4** (CT impulse)**.** *We define the continuous-time (CT) impulse $\delta(\cdot)$ by the property that*

$$\int_{-\infty}^{+\infty} x(t)\,\delta(t)\,dt = x(0)$$

*for all $x(t)$ that are continuous at $t = 0$.*

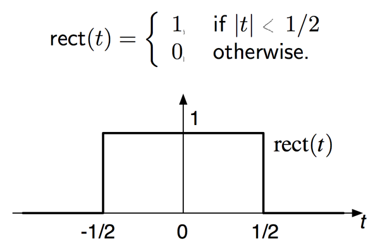



Figure 1: Definition and plot of the unit rectangle.

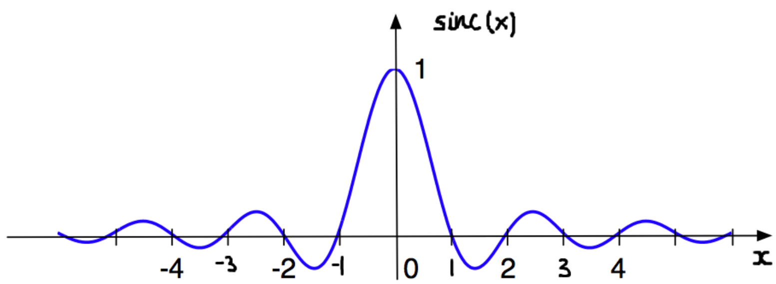



Figure 2: Plot of the sinc function.

## *A Review: Signal Manipulations, CT Convolution, CTFT and Its Properties*

### *Signal manipulations*

PRACTICE EXAMPLES:

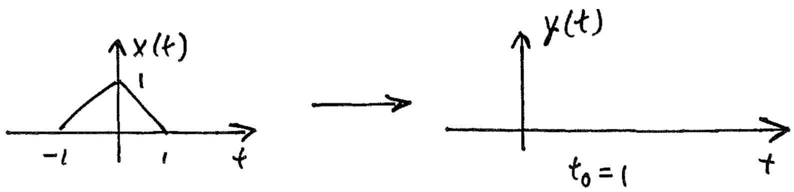



Figure 3: Time shift: $y(t) = x(t - t_0)$. Where does time $t = 0$ move?

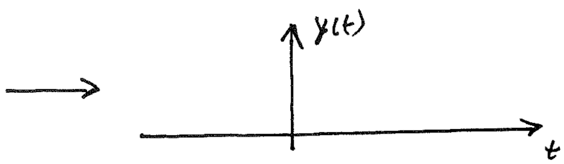



Figure 4: Scaling: $y(t) = x(t/T)$ where $T > 0$.

### *CT convolution*

CT CONVOLUTION IS DEFINED AS

$$x(t) \star h(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t - \tau)\, d\tau.$$

Basic CT linear time-invariant (LTI) systems. The time-shift system $y(t) = x(t - t_0)$ is LTI with impulse response $\delta(t - t_0)$:

$$x(t) \star \delta(t - t_0) = x(t - t_0). \tag{3}$$

EXAMPLE: COMPUTE $y(t) = (x \star h)(t)$ FOR $x(t) = 2\,\mathbb{1}_{(0,2)}(t)$ AND $h(t) = \mathbb{1}_{(0,1)}(t)$.

First sketch $x(t)$ and $h(t)$:

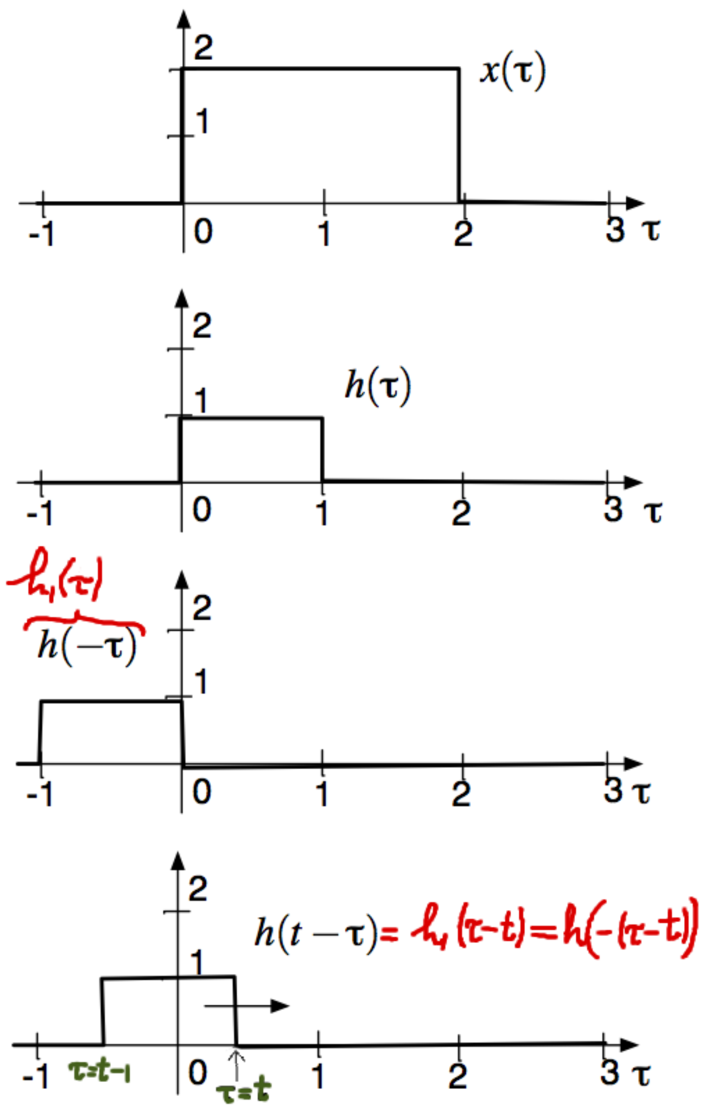
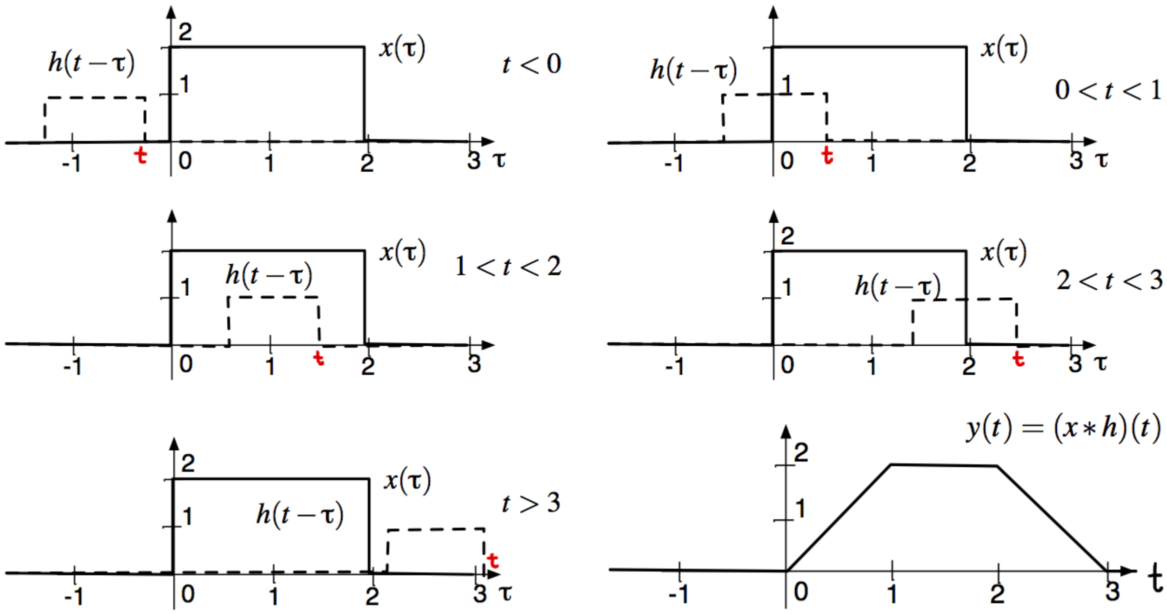



Figure 5: Critical time points: $t - 1 = 0$ and $t = 0$ as well as $t - 1 = 2$ and $t = 2$, i.e. $t = 0, 1, 2, 3$, meaning that we have 5 intervals to consider for $t$.




## *CTFT and its properties*

$X^F(\omega)$ DENOTES CONTINUOUS-TIME FOURIER TRANSFORM (CTFT) OF $x(t)$:

The textbook uses $X(j\omega)$ to denote the CTFT of $x(t)$.

$$X^F(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \tag{4a}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X^F(\omega)\, e^{j\omega t}\, d\omega \tag{4b}$$

where $\omega$ is the frequency in radians per second (rad/s).

Review EE 224 handout `lctftsummary` to solve the practice examples in Fig. 6.

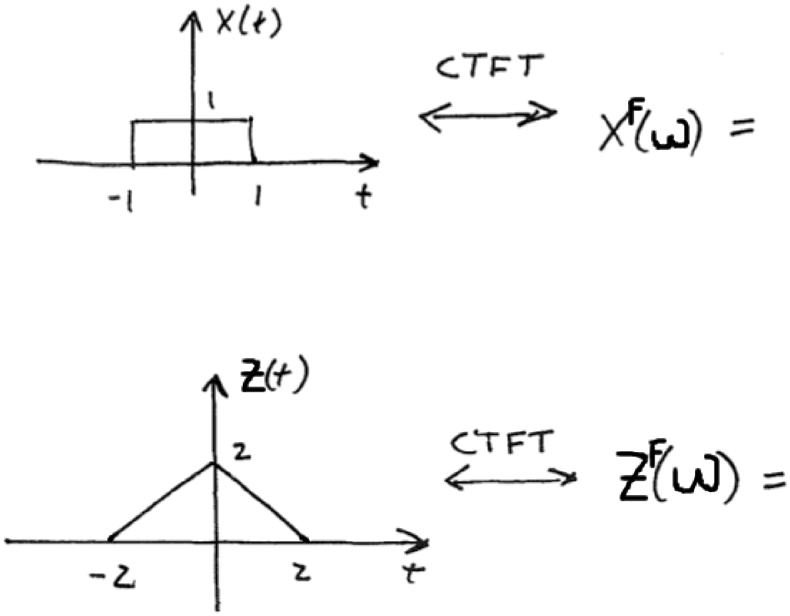



Figure 6: Examples of CTFT properties.

MODULATION PROPERTY: If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X^{\text{F}}(\omega)$, then

$$x(t)\, e^{j\,\omega_0\, t} \overset{\text{CTFT}}{\longleftrightarrow} X^{\text{F}}(\omega - \omega_0) \quad \text{(complex modulation).} \tag{5}$$

GENERALIZED MODULATION PROPERTY. Find CTFT of a signal

$$x(t)\, f(t) \tag{6}$$

where $f(t)$ is periodic with fundamental period $T_0$ and fundamental frequency $\omega_0 = 2\,\pi/T_0$. First, express $f(t)$ using Fourier series (FS):

$$f(t) = \sum_{k=-\infty}^{+\infty} a_k\, e^{j\,k\,\omega_0\, t}$$

and substitute this expansion into (6):

$$x(t) \sum_{k=-\infty}^{+\infty} a_k\, e^{j\,k\,\omega_0\, t} = \sum_{k=-\infty}^{+\infty} a_k\, x(t)\, e^{j\,k\,\omega_0\, t} \overset{\text{CTFT}}{\longleftrightarrow} \sum_{k=-\infty}^{+\infty} a_k\, X^{\text{F}}(\omega - k\,\omega_0). \tag{7}$$

To derive the sampling theorem, we will choose $f(t)$ to be the impulse train, defined in the following.

IDEAL LOWPASS FILTER. The frequency response of the ideal lowpass filter in Fig. 7 can be written as[2]

$$H^{\text{F}}(\omega) = T\, \mathbb{1}_{-\pi/T, \pi/T}(\omega) \tag{8}$$

and the corresponding impulse response $h_{\text{LP}}(t)$ is[3]

$$h(t) = T\, \frac{\pi/T}{\pi}\, \text{sinc}\Big(\frac{\pi/T}{\pi}\, t\Big) = \text{sinc}\Big(\frac{t}{T}\Big). \tag{9}$$

[2] See also Definition 3.

[3] See EE 224 handout `lctftsummary`.

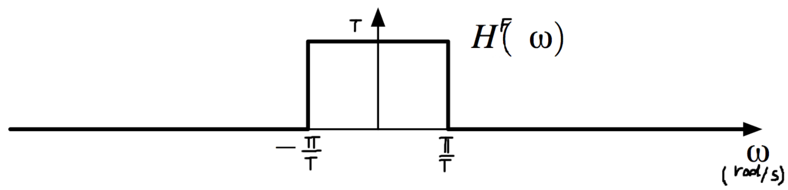



Figure 7: An ideal lowpass filter.

## Poisson Sum Formula

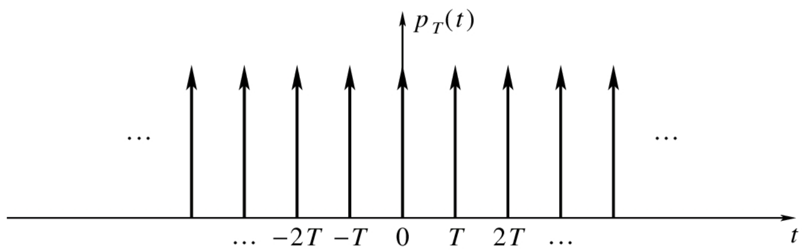



Figure 8: The impulse train $p_T(t)$ is defined as

$$p_T(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

where $T$ denotes its period.

POISSON SUM FORMULA. Consider the Fourier-series representation of the impulse train $p_T(t)$ in Fig. 8:

$$p_T(t) = \sum_{k=-\infty}^{+\infty} a_k \, e^{jk\omega_0 t}$$

where

$$\omega_0 = \frac{2\pi}{T}$$

and

$$a_k = \frac{1}{T} \int_T p_T(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) \, e^{-jk\omega_0 t} \, dt = \frac{1}{T}.$$

Therefore,

$$p_T(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} e^{jk\omega_0 t}. \tag{10}$$

## Sampling

### Introduction

SAMPLING: CONVERSION OF A CONTINUOUS-TIME SIGNAL (USUALLY NOT QUANTIZED) TO A DISCRETE-TIME SIGNAL (USUALLY QUANTIZED).

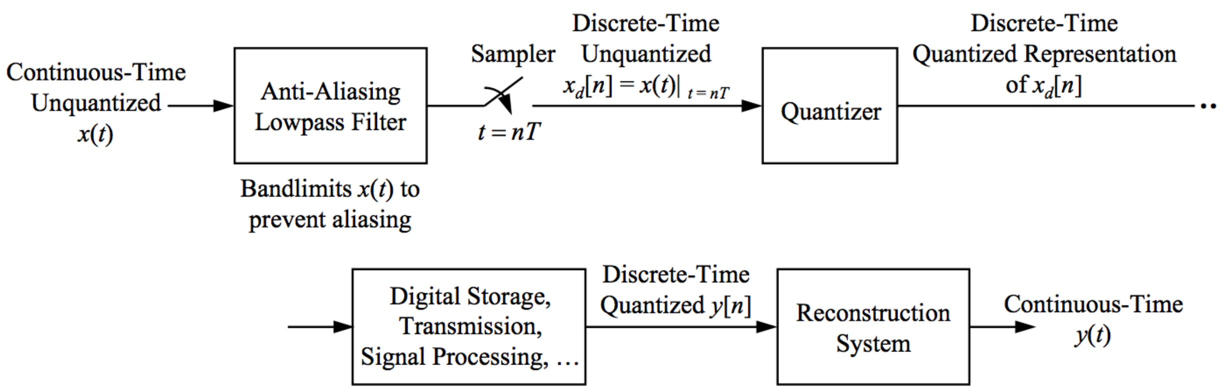
RECONSTRUCTION: CONVERSION OF A DISCRETE-TIME SIGNAL (USUALLY QUANTIZED) TO A CONTINUOUS-TIME SIGNAL.

Why Sample and Reconstruct?

- Digital storage (CD, DVD, etc.)
- Digital transmission (optical fiber, cellular phone, etc.)
- Digital switching (telephone circuit switch, Internet packet switch, etc.)
- Digital signal processing (video compression, speech compression, etc.)
- Digital synthesis (speech, music, etc.).

## *Applications*

HERE IS A TYPICAL SAMPLING AND RECONSTRUCTION SYSTEM:




Quantization causes "noise," limiting the signal-to-noise ratio (SNR) to about 6 dB per bit. We mostly neglect the quantization effects in this class.

## *Point and impulse sampling*

THERE ARE TWO WAYS of looking at the sampled signal: as

1. a sequence of numbers

$$x[n] = x(nT), \; n \text{ integer}$$

*point sampling of $x(t)$*, depicted in Fig. 9 (b), or

2. a continuous-time signal

$$x_P(t) = \sum_{n=-\infty}^{+\infty} x(n\,T)\,\delta(t - n\,T)$$

*impulse sampling of $x(t)$*, depicted in Fig. 9 (c).

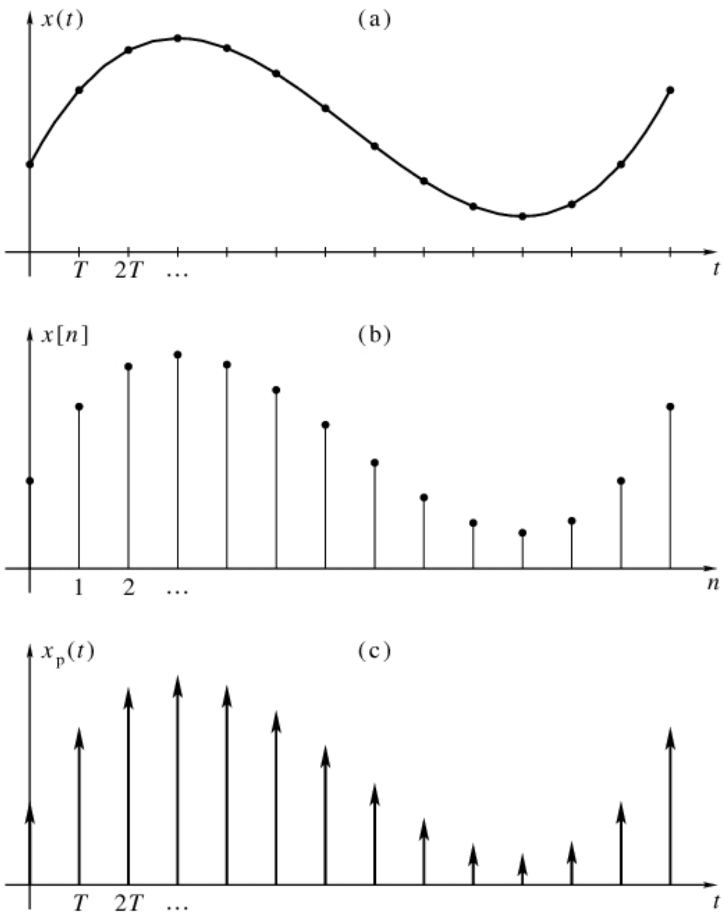
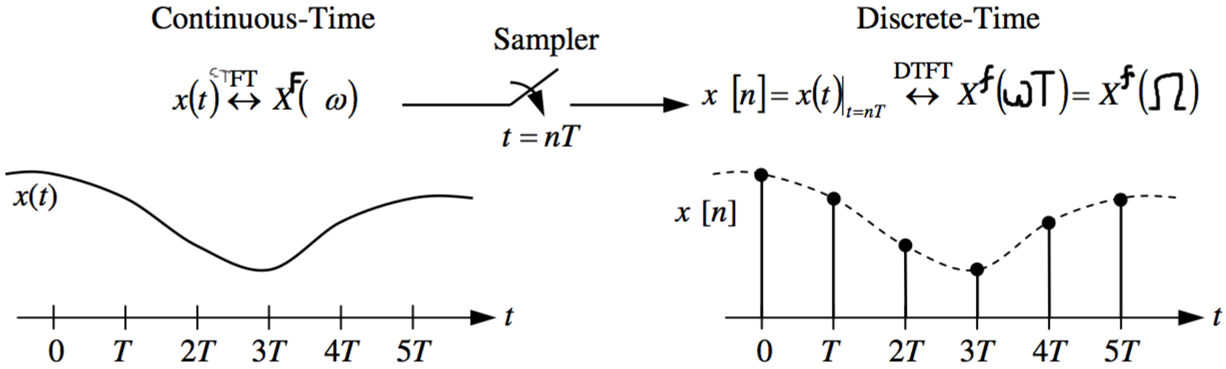



Figure 9: Sampling: (a) CT signal $x(t)$, (b) the point-sampled sequence $x[n]$, and (c) the impulse-sampled signal $x_P(t)$.

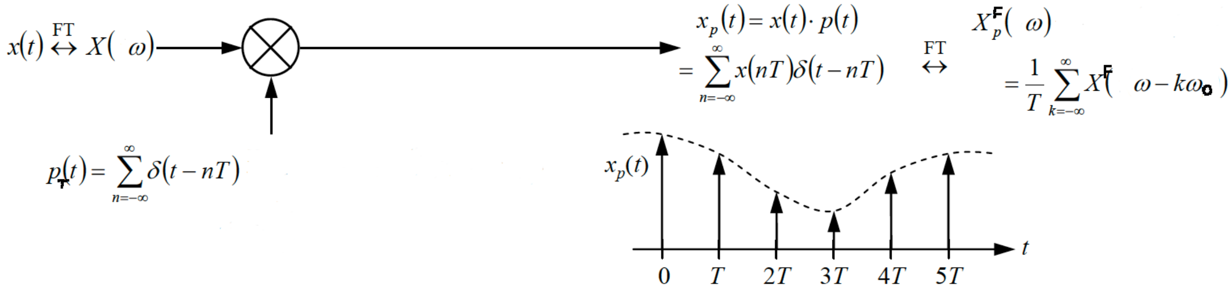
## Point sampling: An actual sampling system mixes continuous and discrete time.




- Continuous-time $x(t)$ specified for all $t$.
- Spectrum $X^{\mathrm{F}}(\omega)$ analyzed by CTFT, frequency variable $\omega$.

- Discrete-time
$$x[n] = x(nT)$$
at $nT$, $n$ integer.
- Spectrum $X^{\mathrm{f}}(\Omega)$ analyzed by DTFT, frequency variable $\Omega = \omega T$.

## Impulse sampling: An equivalent all-CT system.




- "Continuous-time" signal $x_p(t)$ specified for all $t$, but zero except at $t = nT$.
- Spectrum $X_p^{\mathrm{F}}(\omega)$ analyzed using CTFT (which is why we use impulse sampling), with

$$X_p^{\mathrm{F}}(\omega) = X^{\mathrm{f}}(\underbrace{\omega T}_{\Omega}). \tag{11}$$

## *Sampling theorem*

IN THIS HANDOUT, WE FOCUS ON IMPULSE SAMPLING BECAUSE IT REQUIRES ONLY THE KNOWLEDGE OF THEORY OF CT SIGNALS AND CTFT. [4] Recall the impulse train $p_T(t) = \sum_{n=-\infty}^{+\infty} \delta(t - n\,T)$ and define

$$x_P(t) = x(t)\, p_T(t) = \sum_{n=-\infty}^{+\infty} x(t)\,\delta(t - n\,T) = \sum_{n=-\infty}^{+\infty} \underbrace{x(n\,T)}_{x[n]}\,\delta(t - n\,T) \tag{12}$$

which is formally a CT signal.[5] By the Poisson sum formula (10), we have

$$x_P(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T}\, x(t)\, e^{j\,k\,\omega_0\,t}. \tag{13}$$

[4] Since this is a course on digital signal processing, we will turn to DT signals and point sampling starting handout #2. Then, (11) will be the bridge between the CT sampling theory developed in this handout and DT results in the remainder of the class.

[5] However, it is clear that the information it conveys about $x(t)$ is limited to the values $x(n\,T)$, $n$ integer.

Take CTFT of (13):

$$X_P^{\mathrm{F}}(\omega) = \sum_{k=-\infty}^{+\infty} \frac{1}{T}\,\mathrm{CTFT}\{x(t)\, e^{j\,k\,\omega_0\,t}\} = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X^{\mathrm{F}}(\omega - k\,\omega_0) \tag{14}$$

where

$$\omega_0 = \frac{2\,\pi}{T} \quad (\mathrm{rad/s}).$$

For $x(t) \overset{\mathrm{CTFT}}{\longleftrightarrow} X^{\mathrm{F}}(\omega)$ bandlimited to $|\omega| < \omega_{\mathrm{m}}$, we have:

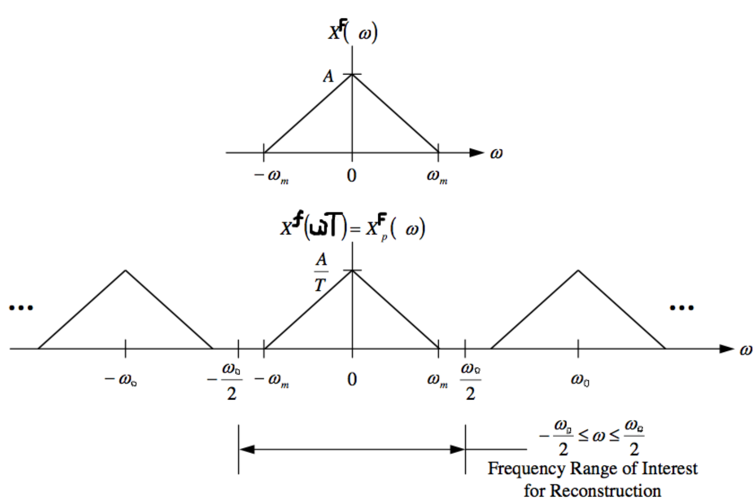



Figure 10: A bandlimited signal spectrum $X^{\mathrm{F}}(\omega)$ and the spectrum $X_P^{\mathrm{F}}(\omega)$ of the corresponding sampled signal.

**Sampling Theorem.** *Suppose* $x(t) \overset{\mathrm{CTFT}}{\longleftrightarrow} X^{\mathrm{F}}(\omega)$ *bandlimited to* $|\omega| < \omega_{\mathrm{m}}$.

- *If the sampling frequency satisfies*[6]

$$\omega_0 > 2\,\omega_{\mathrm{m}} \tag{15}$$

*as in Fig. 10, no aliasing occurs and we can perfectly reconstruct $x(t)$ from its samples*

$$x[n] = x(t)|_{t=n\,T},\ n = 0, \pm 1, \pm 2, \dots$$

*or, equivalently, from $x_P(t)$.*

- *If*

$$\omega_0 \leqslant 2\,\omega_{\mathrm{m}}$$

*aliasing occurs and we cannot reconstruct $x(t)$ perfectly from $x[n]$ in general. (In special cases, we can.)*

[6] (15) is known as the *Nyquist criterion*.

## *Reconstruction*

ASSUME THAT THE NYQUIST REQUIREMENT $\omega_0 > 2\,\omega_{\mathrm{m}}$ IS SATISFIED. We consider two reconstruction schemes:

- ideal reconstruction (with ideal bandlimited interpolation),
- reconstruction with zero-order hold.

### *Ideal Reconstruction: Shannon interpolation formula*

RECALL (14):

$$X_P(t) = \dots + \frac{1}{T} X^{\mathrm{F}}(\omega + \omega_0) + \frac{1}{T} X^{\mathrm{F}}(\omega) + \frac{1}{T} X^{\mathrm{F}}(\omega - \omega_0) + \dots$$

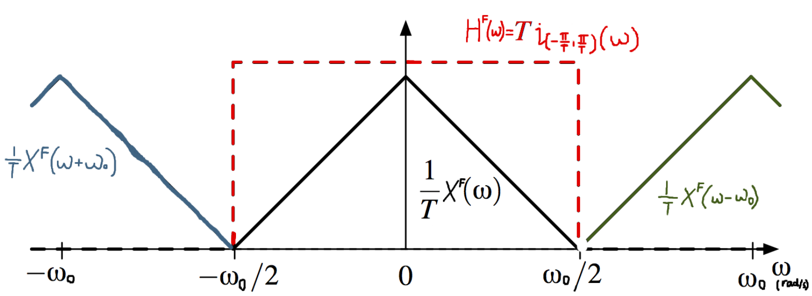



Figure 11: To reconstruct the original CT signal $x(t)$, apply an ideal lowpass filter to the impulse-sampled signal $x_P(t) = x(t)\,p_T(t)$.

Our ideal reconstruction filter has the frequency response:

$$H^{\mathrm{F}}(\omega) = T\,\mathbb{1}_{(-\pi/T, \pi/T)}(\omega)$$

and, consequently, the impulse response [see (9)]

$$h(t) = \text{sinc}\big(\frac{t}{T}\big).$$

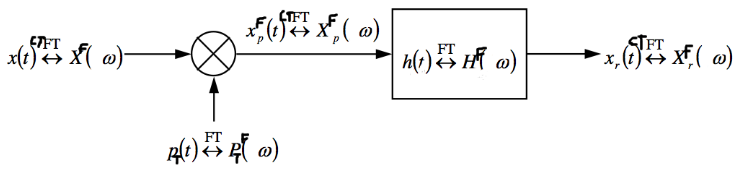
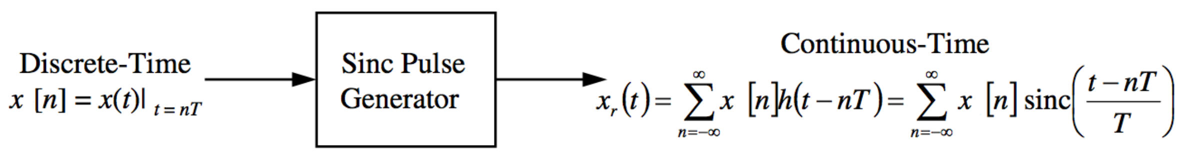



Figure 12: An equivalent all-CT reconstruction system.

Now, the reconstructed signal is

$$x(t) = \underbrace{x_P(t)}_{\text{impulse-sampled signal}} \star h(t) = \sum_{n=-\infty}^{+\infty} x(nT)\underbrace{\delta(t-nT)\star h(t)}_{h(t-nT),\text{ see (3)}} = \sum_{n=-\infty}^{+\infty} x(nT)\,\text{sinc}\Big(\frac{t-nT}{T}\Big)$$

which is the *Shannon interpolation (reconstruction) formula*. The actual reconstruction system mixes continuous and discrete time.




- The reconstructed signal $x_r(t)$ is a train of sinc pulses scaled by the samples $x[n]$.
- This system is difficult to implement because each sinc pulse extends over a long (theoretically infinite) time interval.

## *Ideal reconstruction: Summary*

- Easy to analyze.
- Hard to implement.
- Based on bandlimited sinc pulses.

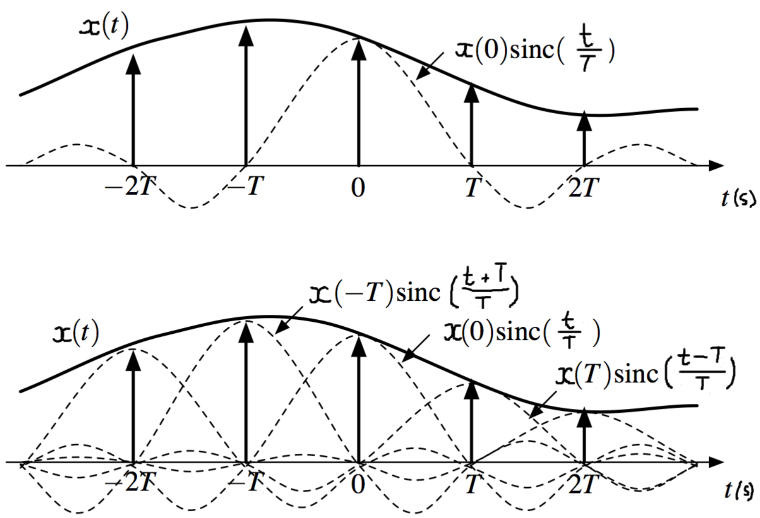



Figure 13: The interpolated signal is a sum of shifted sincs, weighted by the samples $x(nT)$. The sinc function $h(t) = \text{sinc}(t/T)$ shifted to $nT$, i.e. $h(t-T)$, is equal to one at $nT$ and zero at all other samples $lT$, $l \neq n$. The sum of the weighted shifted sincs will agree with all samples $x(nT)$, $n$ integer.

## *A general reconstruction filter*

For the development of the theory, it is handy to consider the impulse-sampled signal $x_P(t)$ and its CTFT.

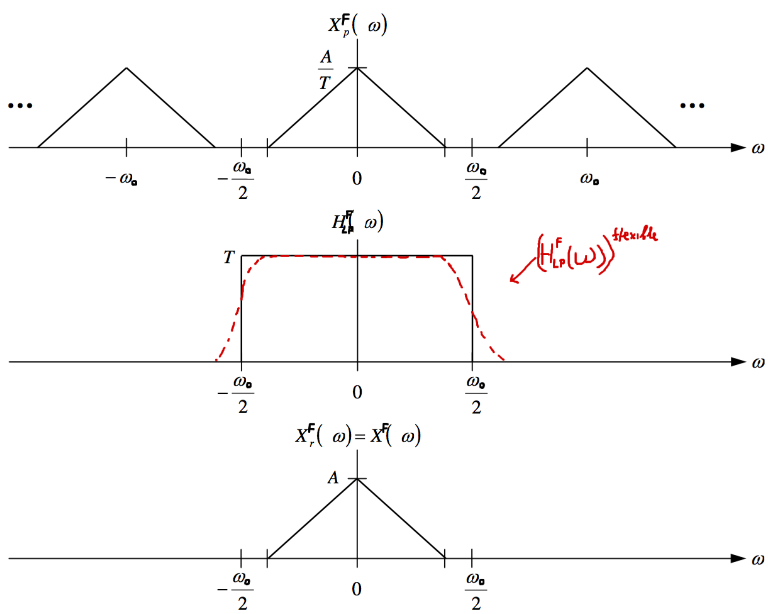



Figure 14: Reconstruction in the frequency domain is lowpass filtering.

$H^{\text{F}}_{\text{LP}}(\omega)$ in Fig. 14 *may not* be a frequency response of an ideal lowpass filter, in contrast with $H^{\text{F}}(\omega)$ in Fig. 11.

Here, the reconstructed signal is $x_r(t)$, with CTFT

$$X_r^{\mathrm{F}}(\omega) = H_{\mathrm{LP}}^{\mathrm{F}}(\omega)\, X_P^{\mathrm{F}}(\omega) \overset{\text{sampling th.}}{=} H_{\mathrm{LP}}^{\mathrm{F}}(\omega)\, \frac{1}{T} \sum_{k=-\infty}^{+\infty} X^{\mathrm{F}}\Big(\omega - \underbrace{\frac{2\,\pi\,k}{T}}_{k\,\omega_0}\Big).$$

NOTE: As sketched in Fig. 14, $h_{\mathrm{LP}}(t) \overset{\mathrm{CTFT}}{\longleftrightarrow} H_{\mathrm{LP}}^{\mathrm{F}}(\omega)$ can be made more flexible than the ideal sinc/boxcar pair; yet, we can still achieve perfect reconstruction. The more we sample above the Nyquist rate, the more flexibility we gain in terms of designing this filter. An example of a more flexible filter is given in Fig. 15.

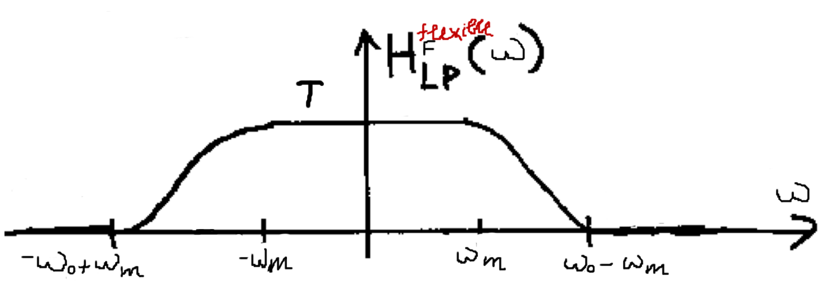



Figure 15: Frequency response of a flexible lowpass reconstruction filter. If $\omega_{\mathrm{m}} = \omega_0/2$, then this frequency response reduces to the standard boxcar frequency response.

## *Reconstruction with zero-order hold*

- MANY PRACTICAL RECONSTRUCTION SYSTEMS use zero-order hold circuits for reconstruction.
- Why? Rectangular pulses are (much) easier to generate than (approximate) sinc pulses.
- Replace the ideal sinc with a rectangular pulse[7]

  $$h_{\mathrm{ZOH}}(t) = \mathrm{rect}\Big(\frac{t - 0.5\,T}{T}\Big)$$

  yielding

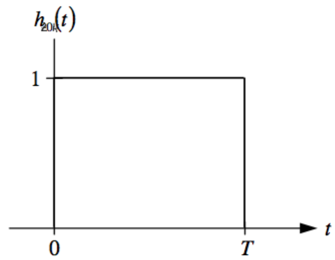
  $$x_{\mathrm{ZOH}}(t) = \sum_{n=-\infty}^{+\infty} x[n]\, h_{\mathrm{ZOH}}(t - n\,T).$$

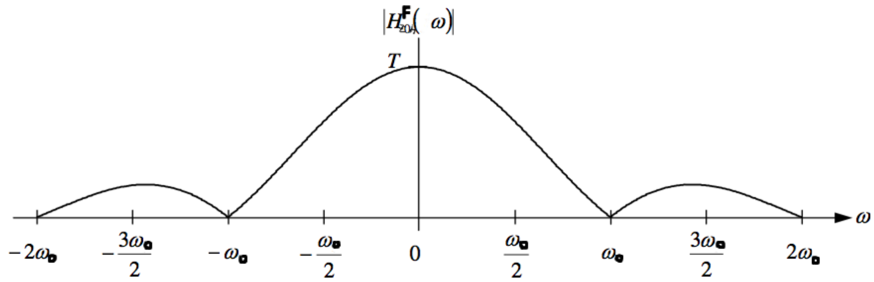
[7] See Definition 1.

Frequency response of the zero-order hold:

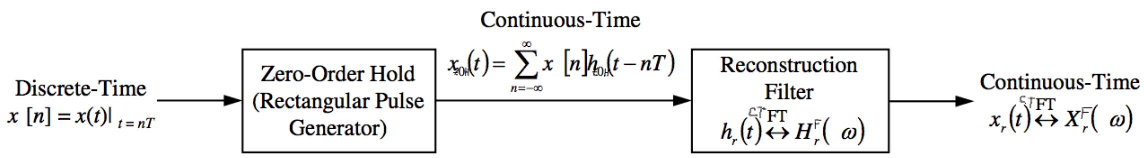
$$H_{\mathrm{ZOH}}^{\mathrm{F}}(\omega) = \int_0^T e^{-j\,\omega\,t}\,dt = \frac{1 - e^{-j\,\omega\,T}}{j\,\omega} = T\,\mathrm{sinc}\Big(\frac{\omega\,T}{2\,\pi}\Big)\, e^{-j\,0.5\,\omega\,T} = T\,\mathrm{sinc}\Big(\frac{\omega}{\omega_0}\Big)\, e^{-j\,\pi\,\frac{\omega}{\omega_0}} \tag{16}$$

recall $\omega_0 = 2\,\pi/T$ and (1).




## Reconstruction system (mixes continuous and discrete time).



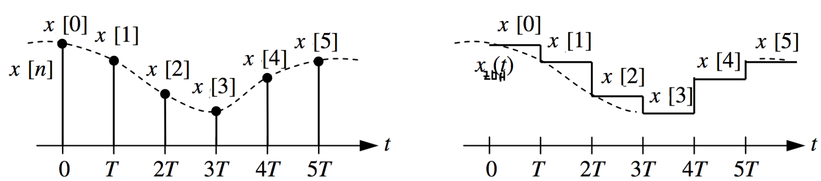




Figure 16: The zero-order hold output $x_{ZOH}(t)$ is a train of rectangular pulses scaled by the samples $x[n]$ (a staircase approximation of $x(t)$), easy to generate.

- Rewrite the zero-order hold output as

$$\begin{aligned}
x_{ZOH}(t) &= \sum_{n=-\infty}^{+\infty} x[n]\, h_{ZOH}(t - n\,T) = \sum_{n=-\infty}^{+\infty} x[n]\, \underbrace{h_{ZOH}(t) \star \delta(t - n\,T)}_{\text{see (3)}} \\
&= h_{ZOH}(t) \star \sum_{n=-\infty}^{+\infty} x[n]\, \delta(t - n\,T) \\
&= h_{ZOH}(t) \star [x(t) \underbrace{\sum_{n=-\infty}^{+\infty} \delta(t - n\,T)}_{p_T(t)}] \\
&= h_{ZOH}(t) \star x_P(t).
\end{aligned}$$

Now, take CTFT of (17):

$$X^F_{ZOH}(\omega) = H^F_{ZOH}(\omega) X^F_P(\omega) \overset{\text{sampling th.}}{=} H^F_{ZOH}(\omega) \frac{1}{T} \sum_{k=-\infty}^{+\infty} X^F(\omega - k\,\omega_0).$$

Finally, the output of the reconstruction filter has the following spectrum [see (16)]:

$$X_r^{\mathrm{F}}(\omega) = H_r^{\mathrm{F}}(\omega)\, X_{\mathrm{ZOH}}^{\mathrm{F}}(\omega) = H_r^{\mathrm{F}}(\omega)\, H_{\mathrm{ZOH}}^{\mathrm{F}}(\omega)\, X_P^{\mathrm{F}}(\omega) = \underbrace{H_r^{\mathrm{F}}(\omega)}_{\substack{\text{reconstruction}\\ \text{filter}}} \quad \underbrace{T \operatorname{sinc}\Big(\frac{\omega}{\omega_0}\Big)\, e^{-j\pi\frac{\omega}{\omega_0}}}_{\substack{\text{sinc with phase factor}\\ \text{from the ZOH circuit}}}\, \underbrace{\frac{1}{T}\sum_{k=-\infty}^{+\infty} X^{\mathrm{F}}(\omega - k\,\omega_0)}_{\substack{\text{shifted copies}\\ \text{from sampling}}}.$$

- We can reconstruct the signal perfectly, i.e.

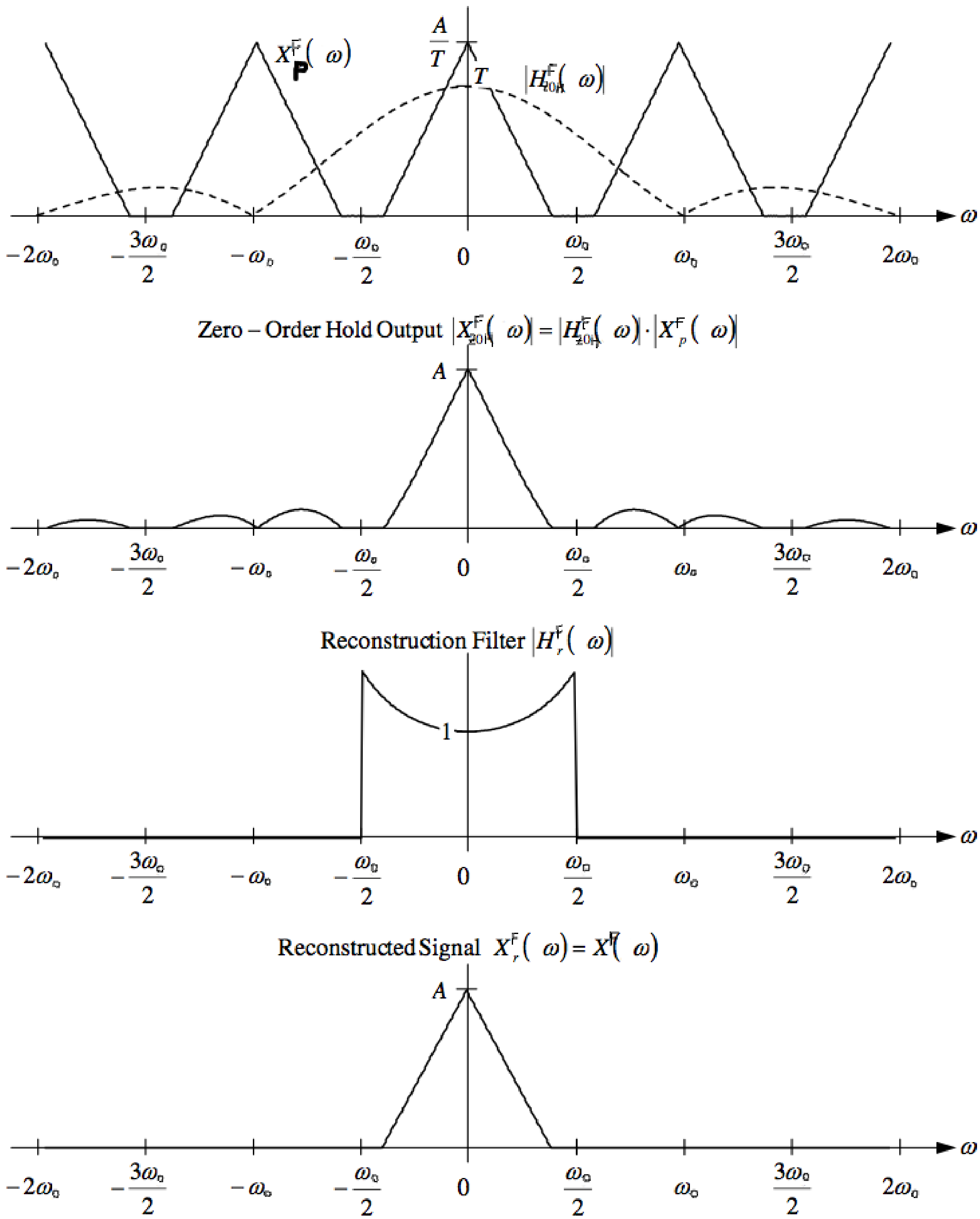
$$x_r(t) = x(t) \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad X_r^{\mathrm{F}}(\omega) = X^{\mathrm{F}}(\omega)$$

  if

  – the Nyquist criterion is satisfied and

  – we can design a reconstruction filter with the following frequency response:

$$H_r^{\mathrm{F}}(\omega) = \underbrace{\frac{e^{j\pi\frac{\omega}{\omega_0}}}{\operatorname{sinc}\Big(\frac{\omega}{\omega_0}\Big)}}_{\substack{\text{compensates ZOH}\\ \text{including delay (hence not causal)}}} \cdot \underbrace{\mathbb{1}_{(-\omega_0/2,\omega_0/2)}(\omega)}_{\substack{\text{removes copies}\\ k \neq 0}}.$$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−ω₀
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0
ω₀/2
ω₀
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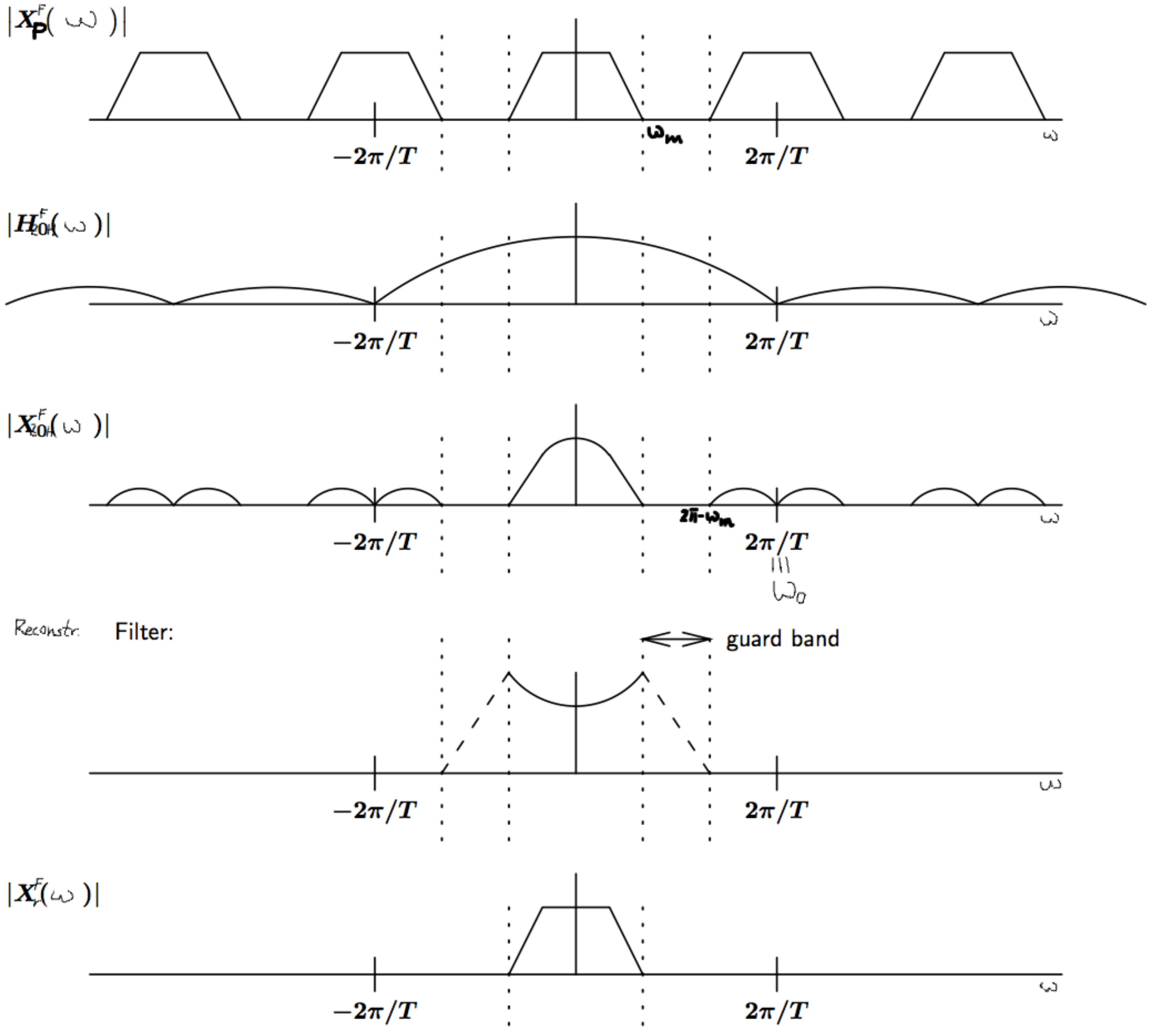
2ω₀
Zero − Order Hold Output
A
Reconstruction Filter
1
Reconstructed Signal
A

We achieve flexibility in designing $H_r^{\mathrm{F}}(\omega)$ by utilizing a sampling rate that is significantly higher than the Nyquist rate, which provides a guard band.

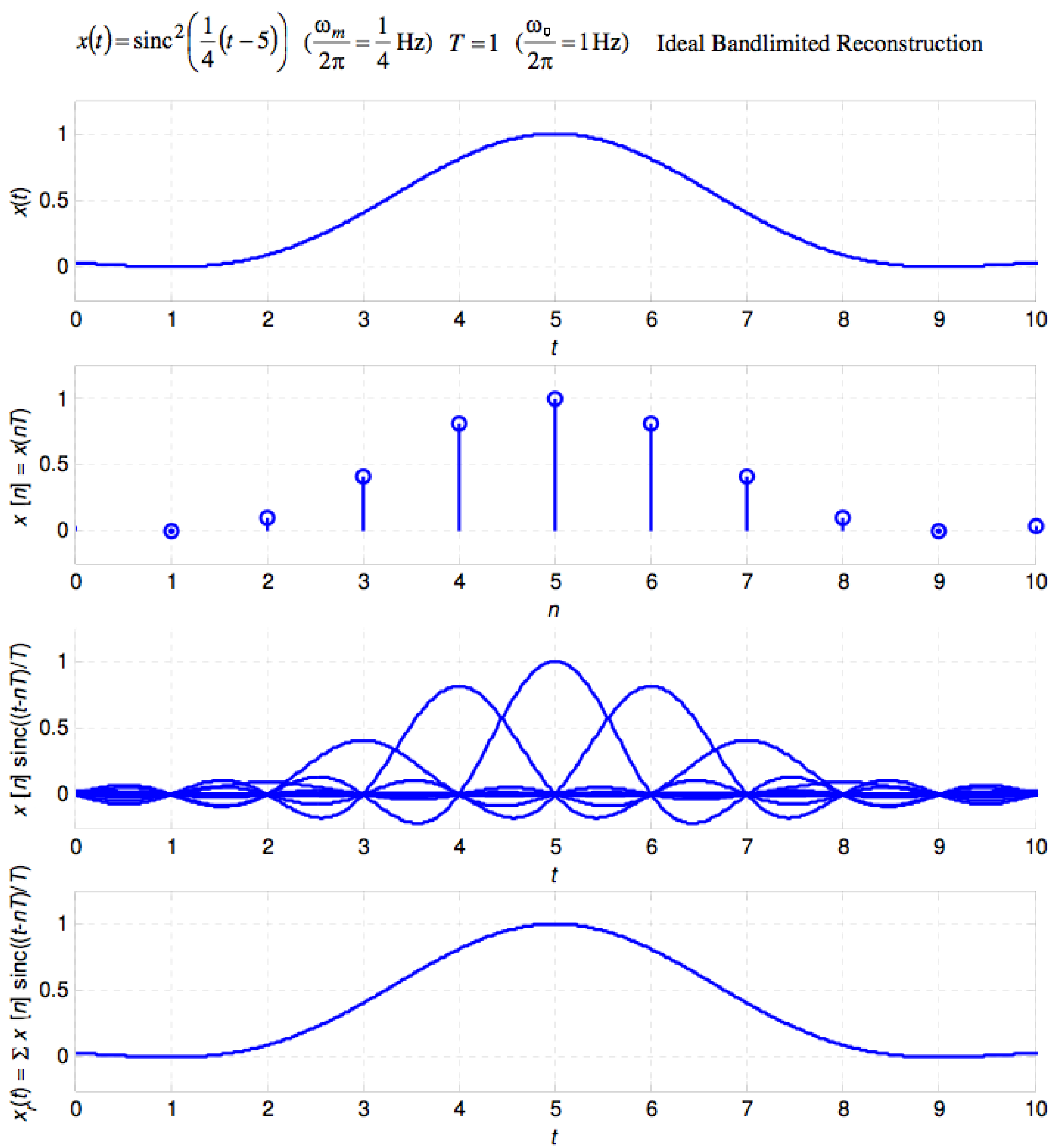



We can boost the sampling rate by digital interpolation — you will see how to do that in Lab 1 and learn the theory later in class.

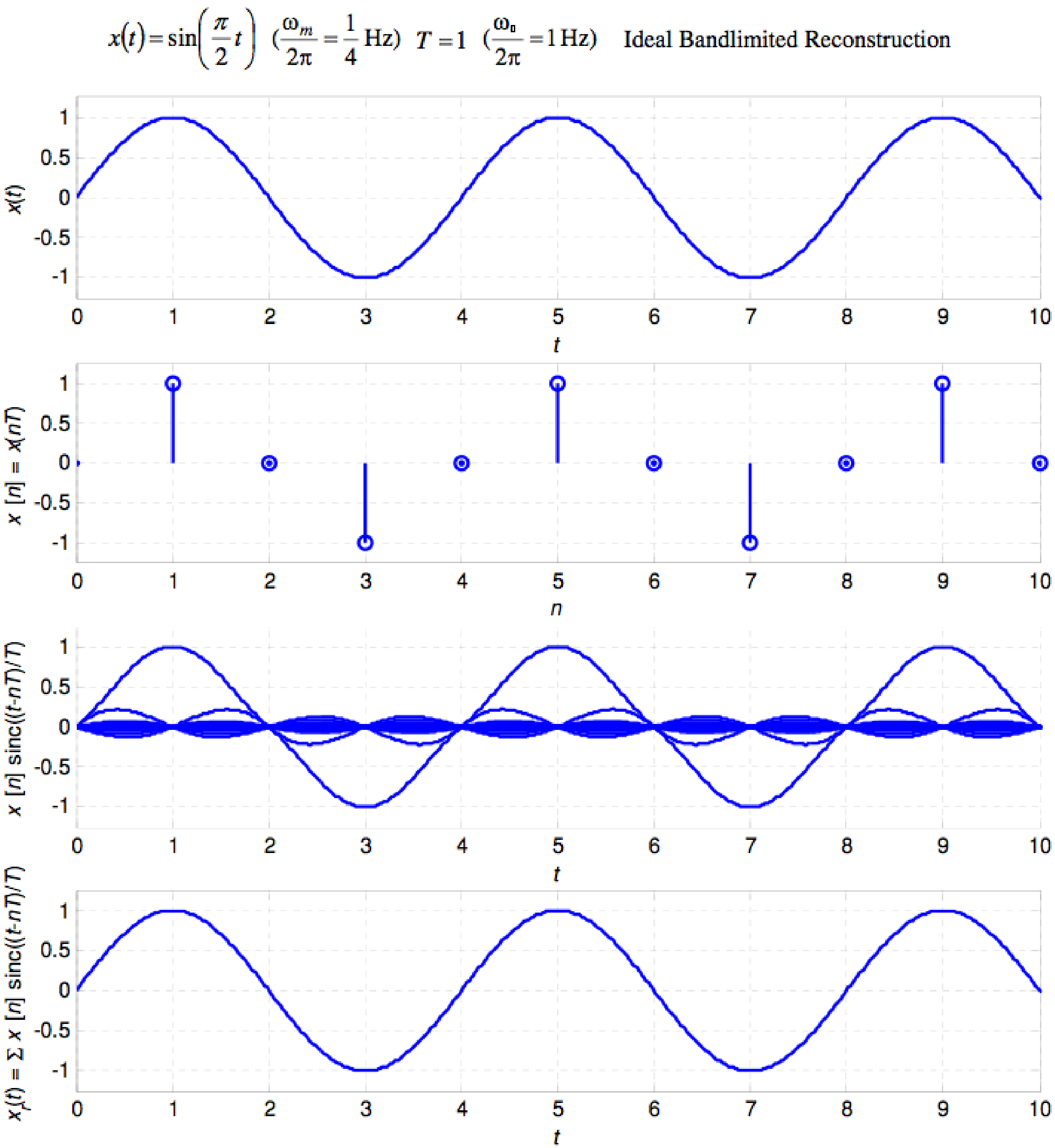
## *Examples of sampling and reconstruction*

IN PRACTICE, we often use one of the standard analog lowpass filters having order 2 to 10 (or so) as reconstruction filters $H_r^{\mathrm{F}}(\omega)$. The last two of the following examples use a second-order analog Butterworth filter with cutoff frequency $\omega_c = \omega_0/2$.

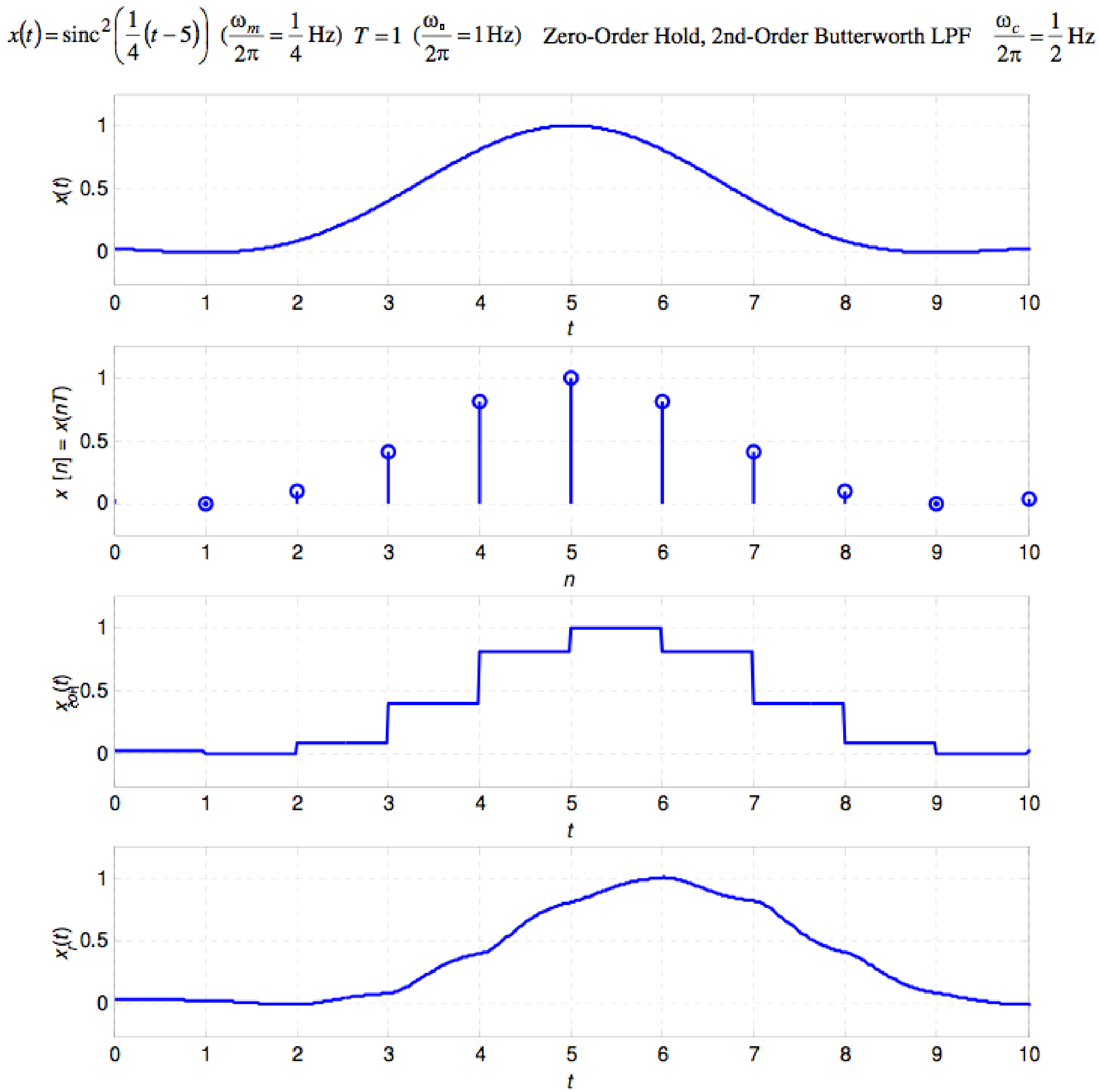
First, recall Fig. 10.



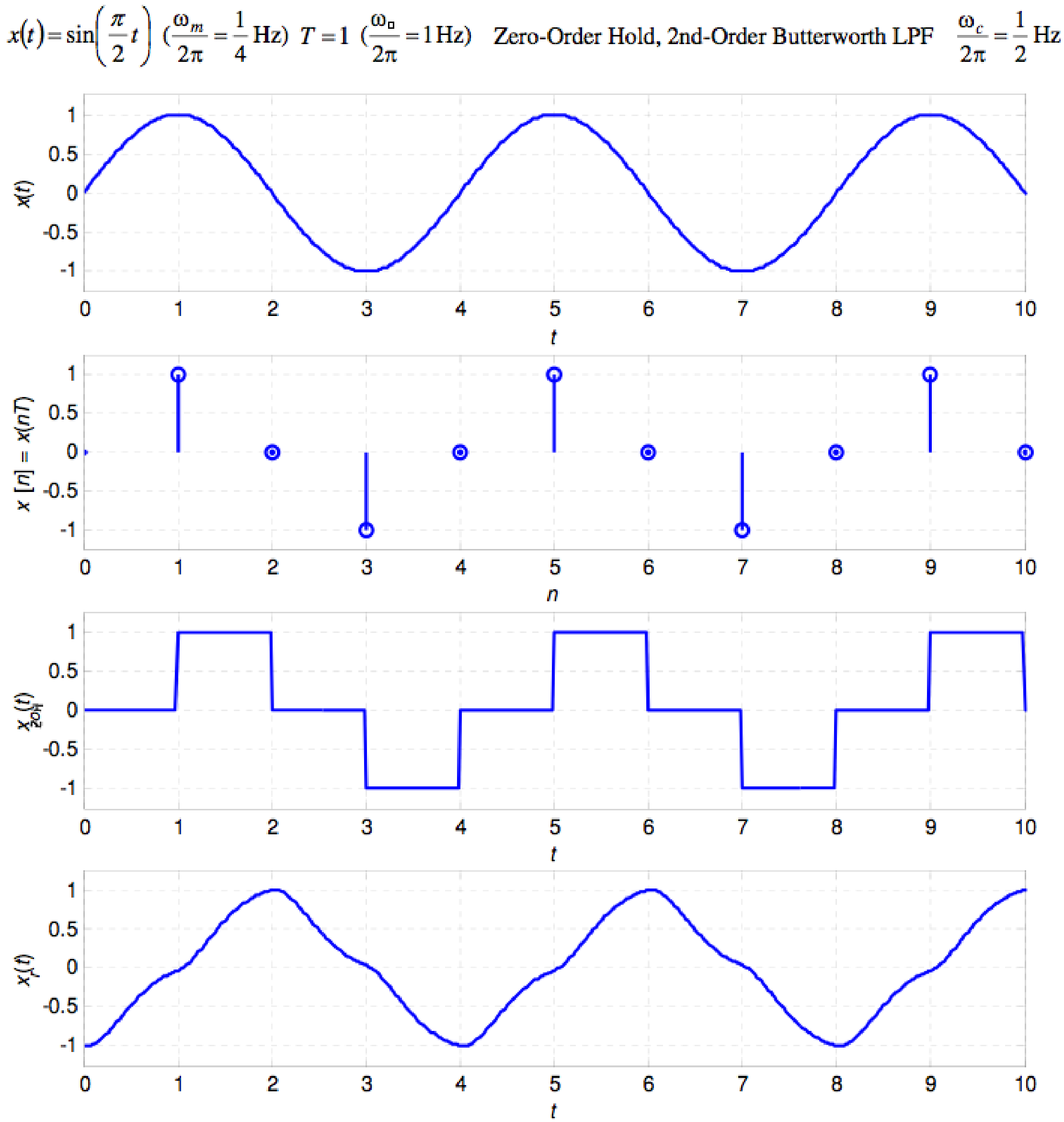
$x(t) = \mathrm{sinc}^2\left(\frac{1}{4}(t-5)\right)$ $(\frac{\omega_m}{2\pi} = \frac{1}{4}\,\mathrm{Hz})$ $T = 1$ $(\frac{\omega_0}{2\pi} = 1\,\mathrm{Hz})$ Ideal Bandlimited Reconstruction
x(t)
t
x [n] = x(nT)
n
x [n] sinc((t-nT)/T)
t
$x_r(t) = \Sigma$ x [n] sinc((t-nT)/T)
t



$x(t) = \sin\left(\frac{\pi}{2} t\right)$ $(\frac{\omega_m}{2\pi} = \frac{1}{4}$ Hz) $T = 1$ $(\frac{\omega_0}{2\pi} = 1$ Hz) Ideal Bandlimited Reconstruction
x(t)
t
x [n] = x(nT)
n
x [n] sinc((t-nT)/T)
t
x_r(t) = Σ x [n] sinc((t-nT)/T)
t



$x(t) = \text{sinc}^2\left(\frac{1}{4}(t-5)\right)$ $\left(\frac{\omega_m}{2\pi} = \frac{1}{4}\text{ Hz}\right)$ $T = 1$ $\left(\frac{\omega_s}{2\pi} = 1\text{ Hz}\right)$ Zero-Order Hold, 2nd-Order Butterworth LPF $\frac{\omega_c}{2\pi} = \frac{1}{2}$ Hz
x(t)
t
x [n] = x(nT)
n
$x_{ZOH}(t)$
t
$x_r(t)$
t



$x(t) = \sin\left(\frac{\pi}{2}t\right)$ $(\frac{\omega_m}{2\pi} = \frac{1}{4}$ Hz) $T = 1$ $(\frac{\omega_s}{2\pi} = 1$ Hz) Zero-Order Hold, 2nd-Order Butterworth LPF $\frac{\omega_c}{2\pi} = \frac{1}{2}$ Hz
$x(t)$
$t$
$x[n] = x(nT)$
$n$
$x_{zoh}(t)$
$t$
$x_r(t)$
$t$

## *Comments on Lab 1*

### *Sampling part of Lab 1*

BASIC FACT: A BANDLIMITED SIGNAL WITH BANDWIDTH $f_m$ (IN Hz) can be reconstructed perfectly from its samples if the sampling rate $f_0 = 1/T$ is twice the signal bandwidth (or more): $f_0 \geqslant 2f_m$.

Typically, we think of sampled sinusoids as looking like that in Fig. 17.

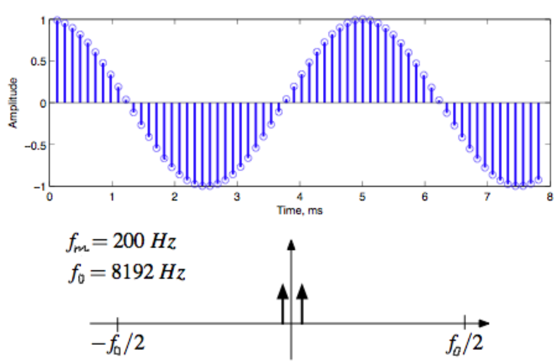



Figure 17: Sampled sinusoid. At this sampling rate, it is easy to believe that we can reconstruct the sinusoid from its samples.

Most sampled sinusoids are much less recognizable:

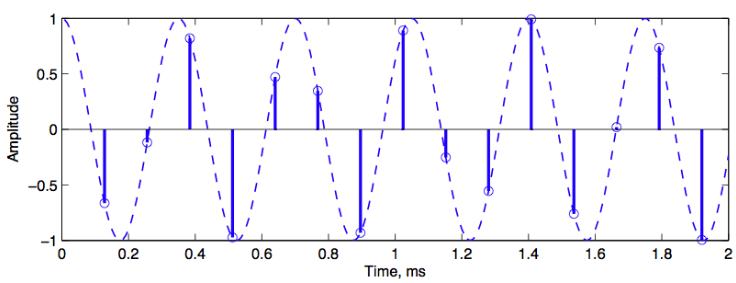



Figure 18: Sinusoid sampled at a much lower sampling rate.

CONCLUSION: The fact that the signal was bandlimited before sampling is a very powerful constraint in the reconstruction of the continuous-time signal.

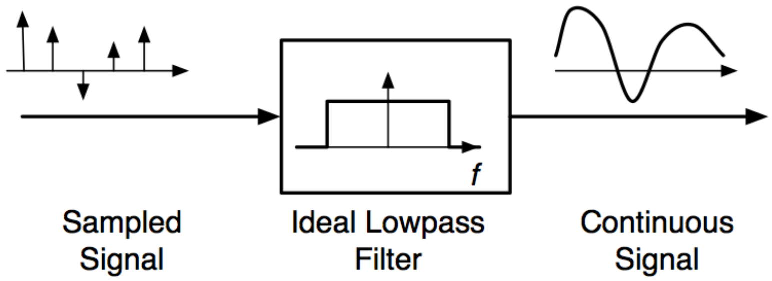



Figure 19: Continuous-time model of the reconstruction of a discrete-time signal.

## *Reconstruction part of Lab 1*

HOW IMPORTANT IS THE LOWPASS FILTER RESPONSE OF THE RECONSTRUCTION FILTER IN FIG. 19? You will look at the improvement in reconstruction as you go from a very simple lowpass filter to higher-performance lowpass filters.

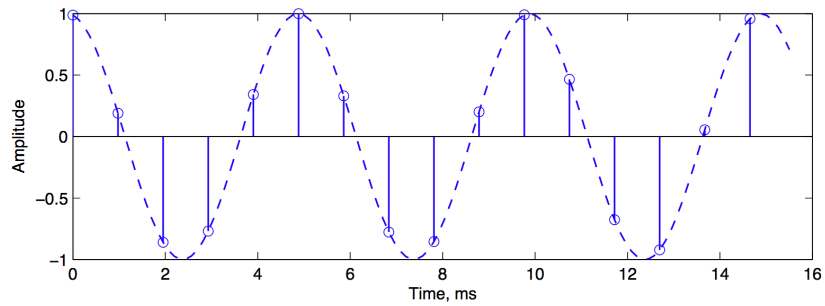
BASIC PROBLEM: You have one second of a 200 Hz sinusoid, sampled at 1024 Hz. You want to reconstruct it as accurately as possible. Since everything in MATLAB is inherently discrete time, we will consider a closely related problem.

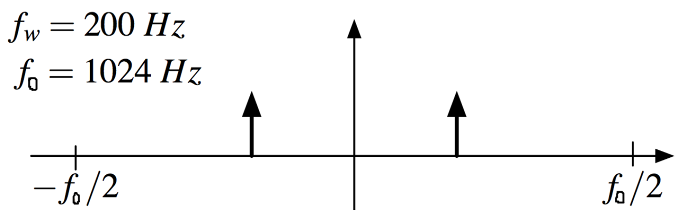
- We start with a 200 Hz sinusoid sampled at 8192 Hz.
- If we take every eighth sample (*subsampling*, or *decimating* by a factor of eight), we have the 200 Hz sinusoid sampled at 1024 Hz.
- We then wish to recover the 7/8ths of the samples we threw away.

Conceptually, the 8192 Hz sampling rate is so high that we can consider the sampled 200 Hz sinusoid to be continuous.

The 8192 Hz sampling rate was chosen so that the signals would all be in the audio range. This is the sampling rate that MATLAB assumes for sound — you can play and hear the reconstructions.

The first 16 ms of the 1024 Hz sampled signal look like this:




This is sampled well above the Nyquist rate, which is 400 Hz. Simple interpolation methods will not be adequate.

### *Lowpass reconstruction filters*

ONE-SAMPLE ZERO-ORDER HOLD:

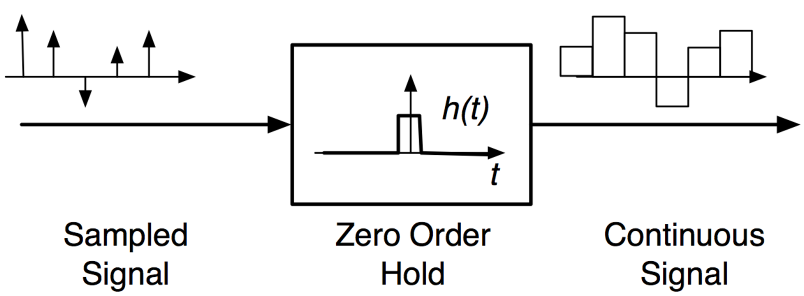



Figure 20: Convolution with a one-sample wide (at 1024 Hz) rect() function.

Common approach, often followed by an additional reconstruction filter $H_r^{\mathrm{F}}(\omega)$ to correct for the passband frequency response of the rect() and suppress sidelobes at multiples of $\omega_0$ (in rad/s), see the earlier discussion in this handout.

LINEAR INTERPOLATION:

This has better suppression of the sidelobes and more passband distortion than the rect().

Ideally, we wish to use the perfect filter with a sinc() impulse response. This is not practical, so instead we approximate the infinite-duration sinc by a segment that we extract with a window function.

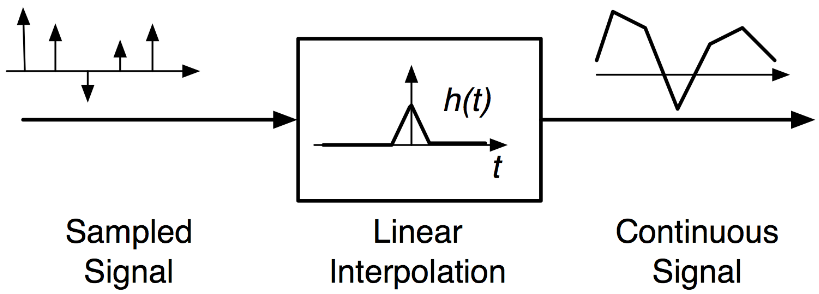
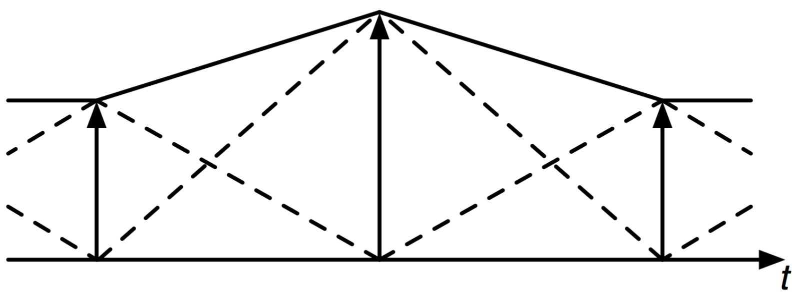



Figure 21: Convolution with a two-sample wide (at 1024 Hz) wedge() function.



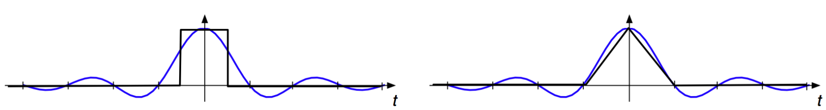



Figure 22: The rect() and wedge() filters are zero- and first-order approximations to the sinc.

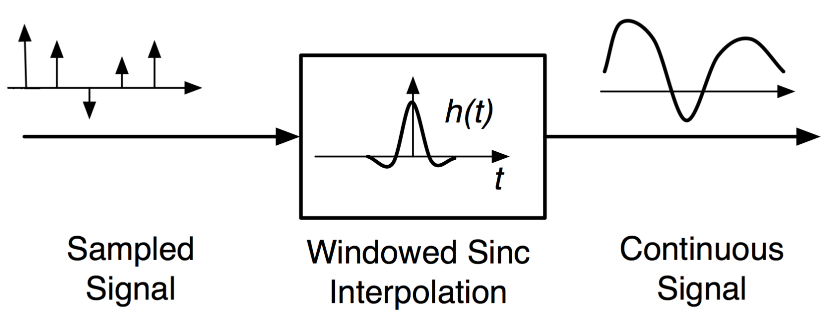



Figure 23: Approximate interpolation: Convolution with a windowed sinc.

Windowed sinc:

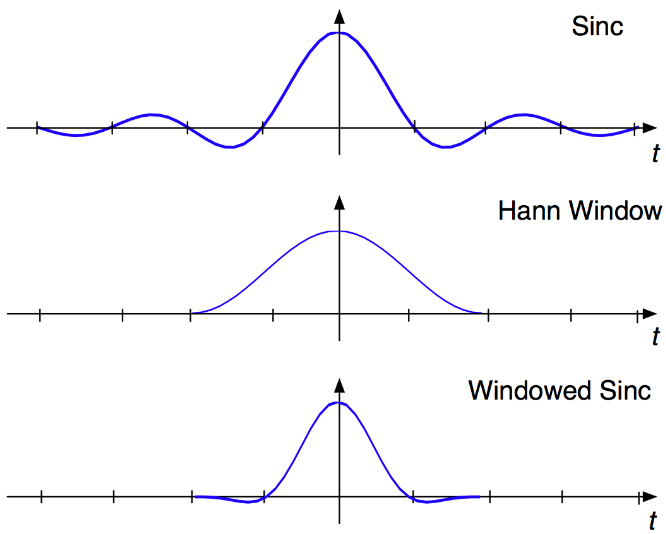



Figure 24: First case: A 4-sample windowed sinc (at 1024 Hz sampling).

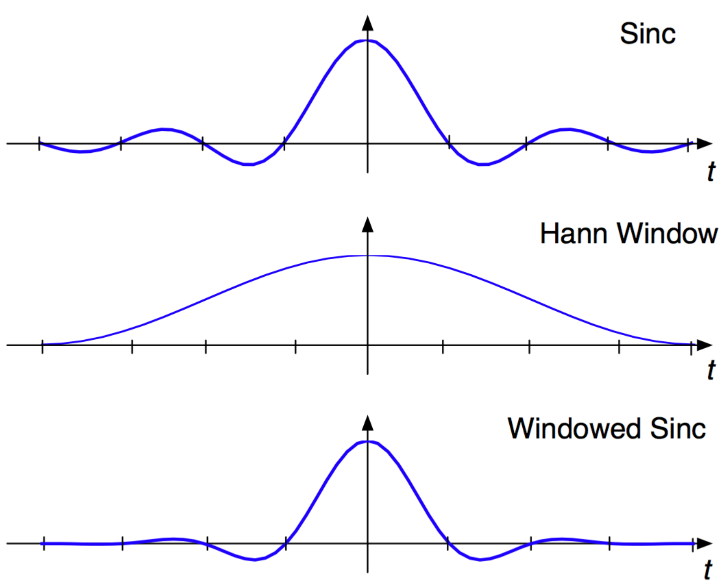
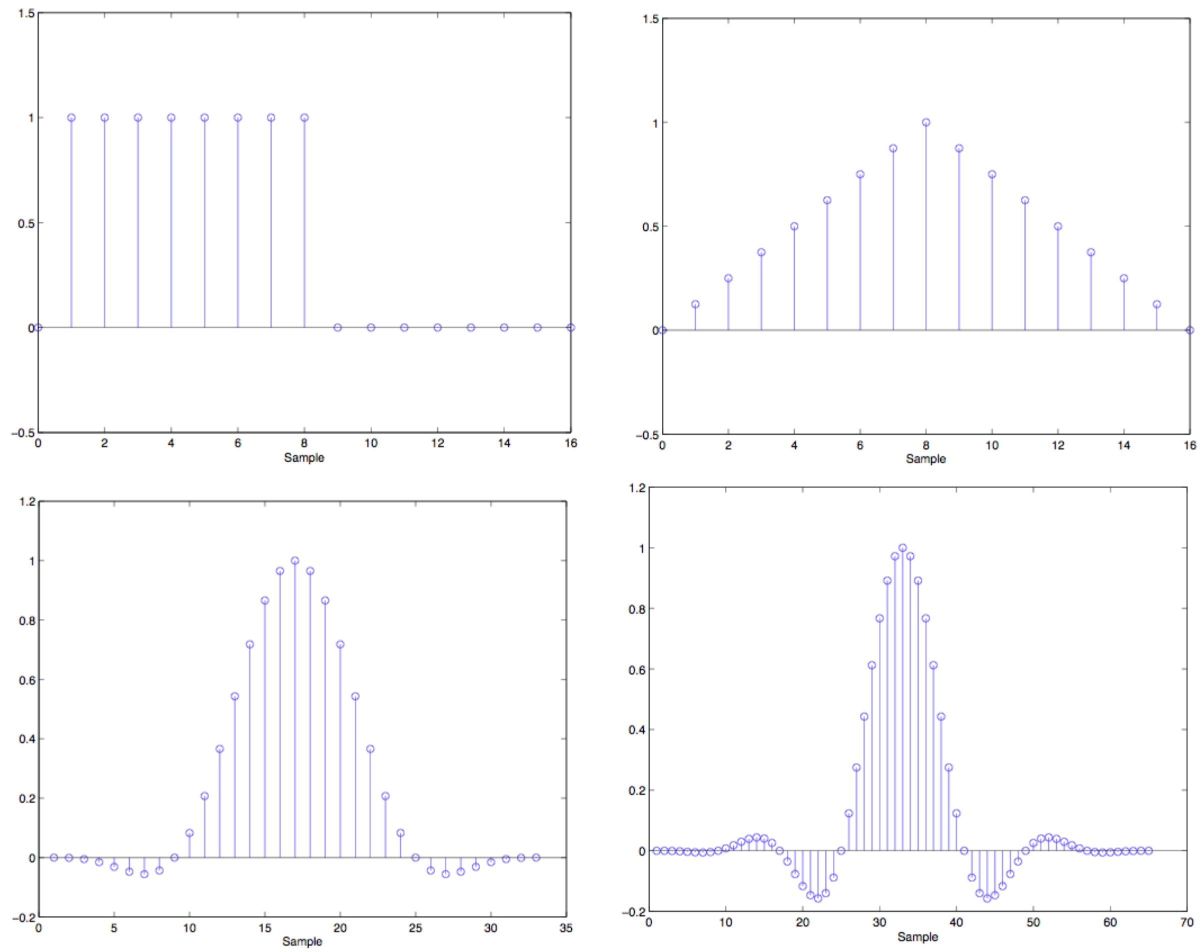


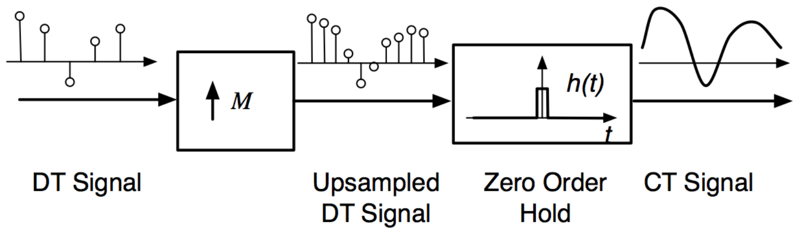

Figure 25: Second case: An 8-sample windowed sinc (at 1024 Hz sampling).

## *DT lowpass reconstruction filters*

IN LAB 1, WE WILL DO THE FILTERING IN DISCRETE TIME using sampled versions of the filters, and the convolution sum.




What we actually do here is *upsampling* or *discrete-time interpolation*: the sampling rate is increased by a factor of $M$ in discrete time, in order to reduce the demands of the D/A conversion. This allows us to use a very simple D/A converter. We will come back to this later at the end of semester.




This is commonly done in CD players, where the data sampling rate is 44.1 kHz. This rate is upsampled by a factor of 8 to 352.8 kHz. By doing so, the need for correction of the ZOH passband distortion is effectively eliminated.

## *References*

A. V. Oppenheim and A. S. Willsky. *Signals & Systems*. Prentice Hall, Upper Saddle River, NJ, 1997.