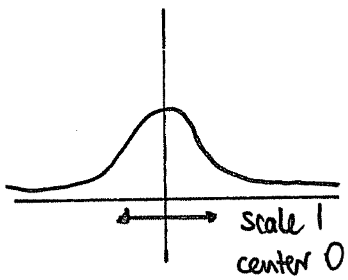
Multi-Resolution Approximation

$$V_0 = \text{span}\{\varphi(x-j) : j = -\infty..\infty\}$$

$\varphi(x) =$ scaling function

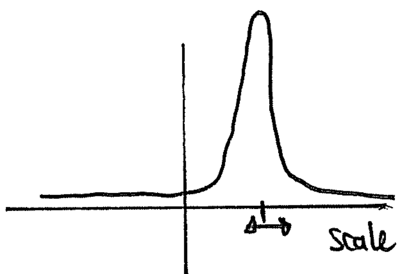
$$\langle \varphi(x), \varphi(x-j) \rangle = \delta_{0j} = \begin{cases} 1 : & j=0 \\ 0 : & j \neq 0 \end{cases}$$

$$\varphi_j^k(x) = 2^{k/2}\varphi(2^k x - j)$$



$\varphi(x)$

scale 1
center 0



$\varphi_3^2(x)$

scale $2^{-2} = 1/4$
center $3 \cdot \frac{1}{4} = 3/4$

$$V_k = \text{span}\{\varphi_j^k(x) : j = -\infty..\infty\}$$

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots$$

with $\overline{\bigcup_k V_k} = L^2$

$$\bigcap_k V_k = \{0\}$$

$P_k f(x) = \perp$ projection of $f \in L^2$ onto V_k

$$= \sum_j \langle f, \varphi_j^k \rangle \varphi_j^k(x)$$

interpreted as approximation to f at resolution 2^{-k}.

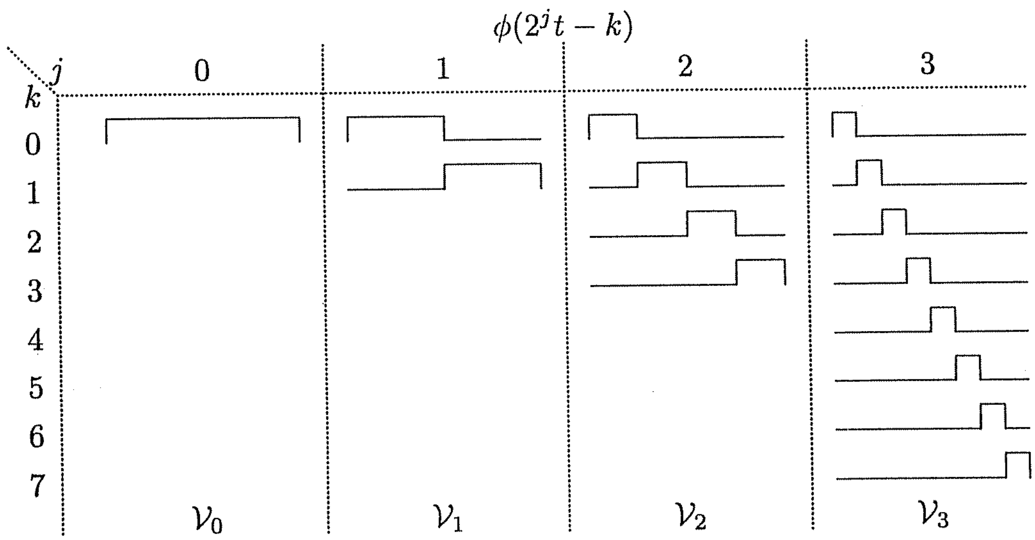
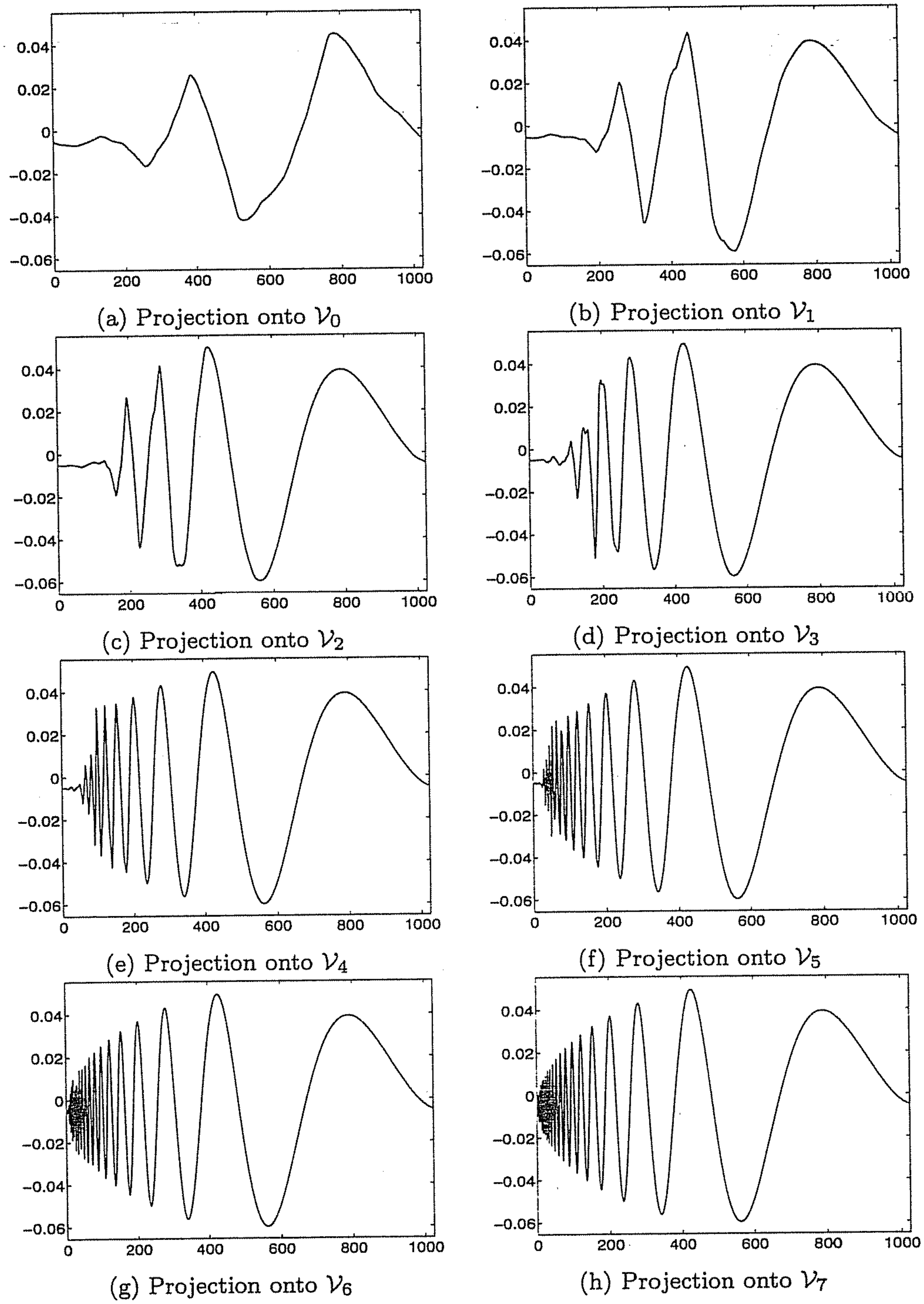


Figure 2.11. Haar Scaling Functions that Span $\mathcal{V}_j$



(a) Projection onto $\mathcal{V}_0$ (b) Projection onto $\mathcal{V}_1$

(c) Projection onto $\mathcal{V}_2$ (d) Projection onto $\mathcal{V}_3$

(e) Projection onto $\mathcal{V}_4$ (f) Projection onto $\mathcal{V}_5$

(g) Projection onto $\mathcal{V}_6$ (h) Projection onto $\mathcal{V}_7$

Figure 2.9. Projection of the Doppler Signal onto $\mathcal{V}$ Spaces using $\phi_{D8'}$

$W_0 = \perp$ complement of V_0 in V_1

$$V_0 \oplus W_0 = V_1$$

Usually

$$W_0 = \text{span}\{\psi(x-j) : j = -\infty..\infty\}$$

$$\langle \psi(x), \psi(x-j) \rangle = \delta_{0j}$$

$$\langle \psi(x), \varphi(x-j) \rangle = 0$$

$\psi(x) =$ mother wavelet

Then

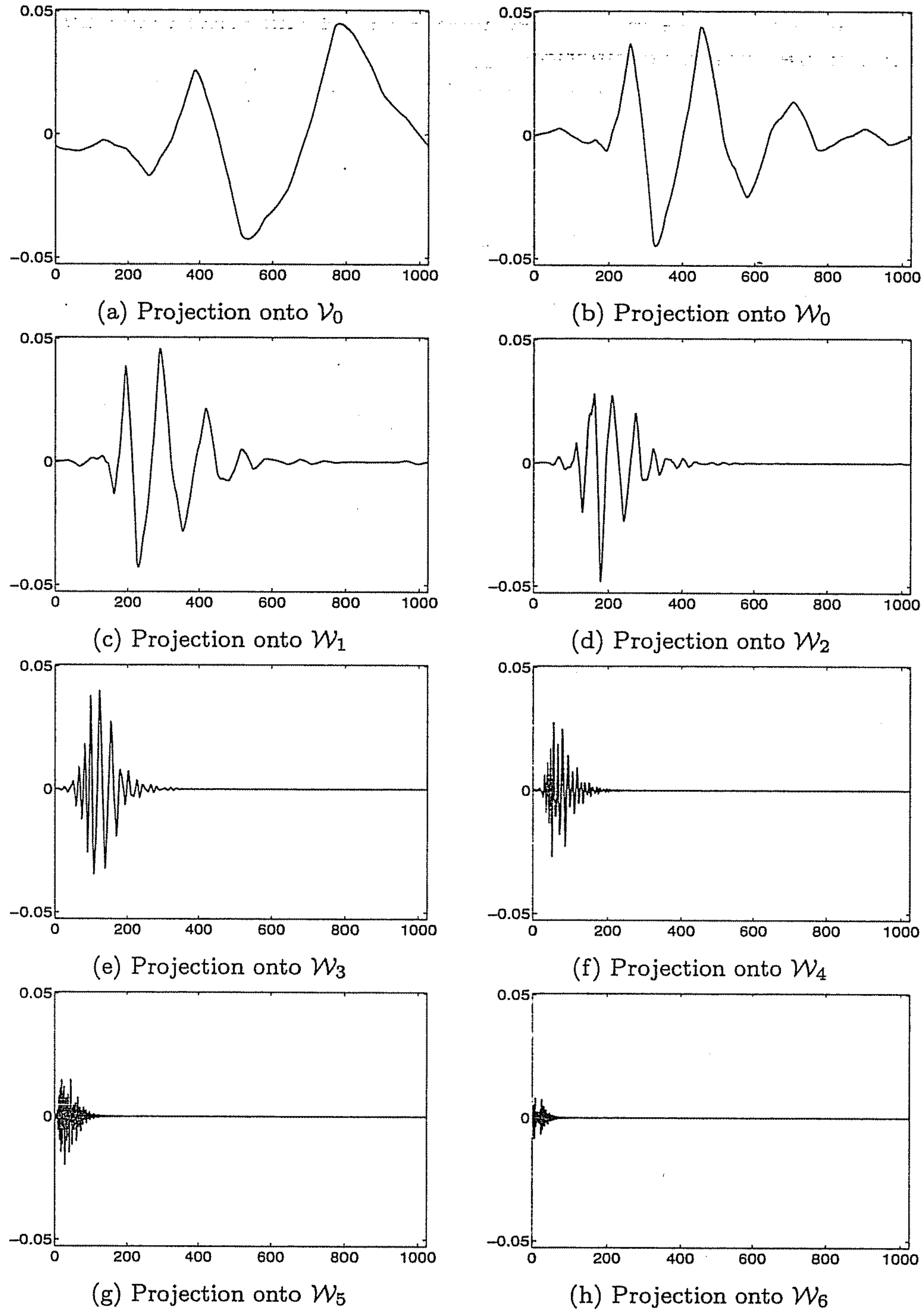
$$V_k \oplus W_k = V_{k+1}$$

$$W_k = \text{span}\{\psi_j^k(x) : j = -\infty..\infty\}$$

$Q_k f(x) = \perp$ projection of f onto W_k

$$= \sum_j \langle f, \psi_j^k \rangle \psi_j^k(x)$$

interpreted as fine detail in f at resolution 2^{-k}.



(a) Projection onto $\mathcal{V}_0$

(b) Projection onto $\mathcal{W}_0$

(c) Projection onto $\mathcal{W}_1$

(d) Projection onto $\mathcal{W}_2$

(e) Projection onto $\mathcal{W}_3$

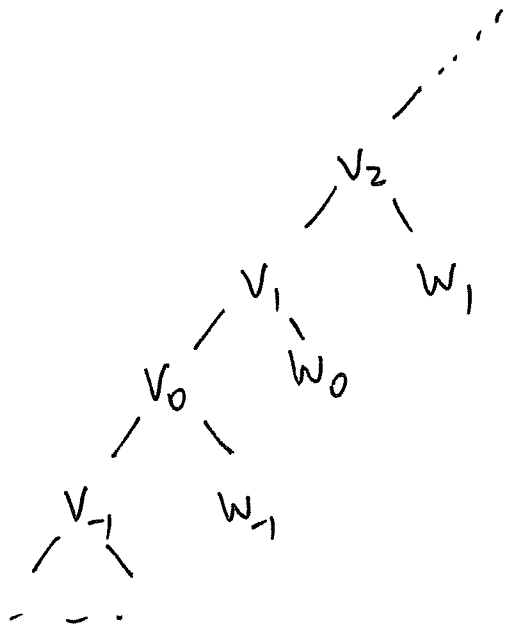
(f) Projection onto $\mathcal{W}_4$

(g) Projection onto $\mathcal{W}_5$

(h) Projection onto $\mathcal{W}_6$

Figure 2.10. Projection of the Doppler Signal onto $\mathcal{W}$ Spaces using $\psi_{D8'}$

Infinite Setting



$$V_j \subset V_{j+1}$$

$$W_j \perp W_k \qquad j \neq k$$

$$\overline{\bigoplus W_k} = L^2$$

$$f(x) = \sum_k Q_k f(x) = \sum_k \sum_j \langle f, \psi_j^k \rangle \psi_j^k(x)$$

Finite Setting

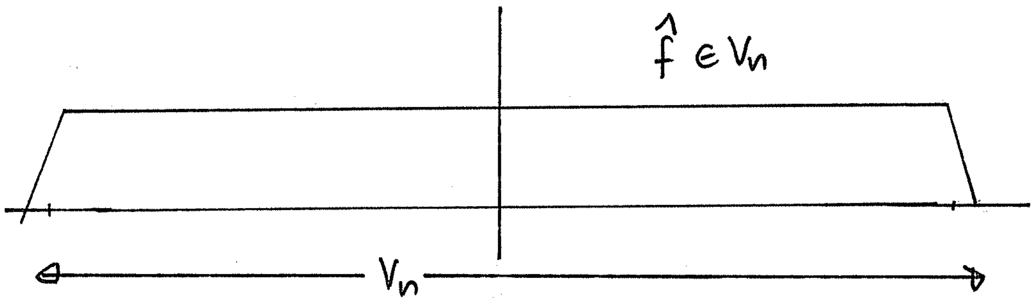
$$
\begin{tikzcd}[column sep=small]
 & & & V_n \arrow[dl, dash] \arrow[dr, dash] & \\
 & & V_{n-1} \arrow[dl, dash] \arrow[dr, dash] & & W_{n-1} \\
 & \vdots & & & \\
 & V_{n-l+1} \arrow[dl, dash] \arrow[dr, dash] & & & \\
V_{n-l} & & W_{n-l} & &
\end{tikzcd}
$$

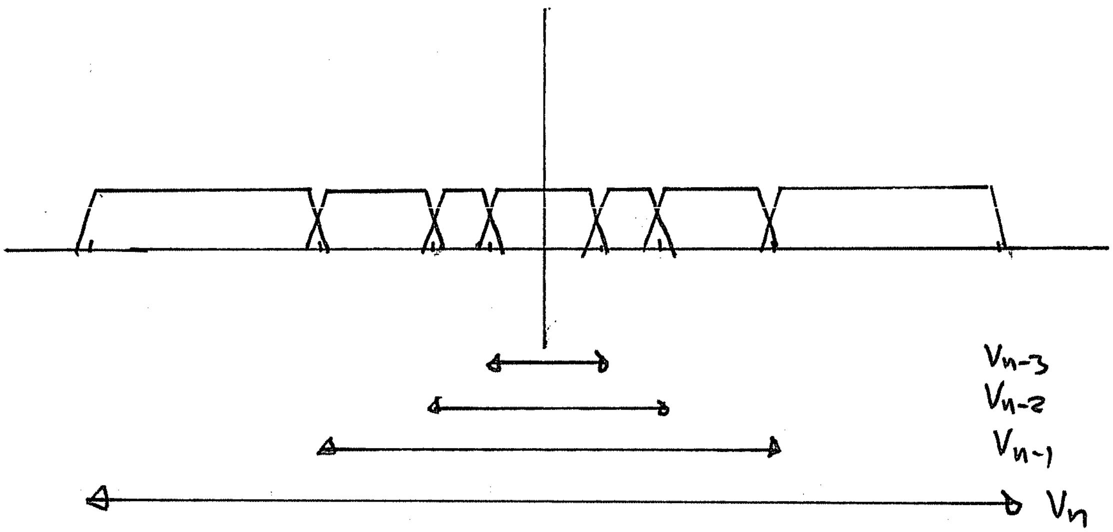
assume $f \in V_n$ (or start with $P_n f$)

Then
$$
\begin{aligned}
f &= P_{n-1} f \oplus Q_{n-1} f \\
&= P_{n-2} f \oplus Q_{n-2} f \oplus Q_{n-1} f \\
&= \cdots \\
&= P_{n-l} f \oplus Q_{n-l} f \oplus \cdots \oplus Q_{n-1} f
\end{aligned}
$$

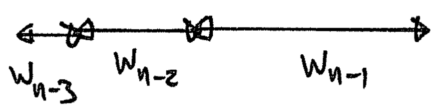
wavelet decomposition of f through l levels

Frequency Picture

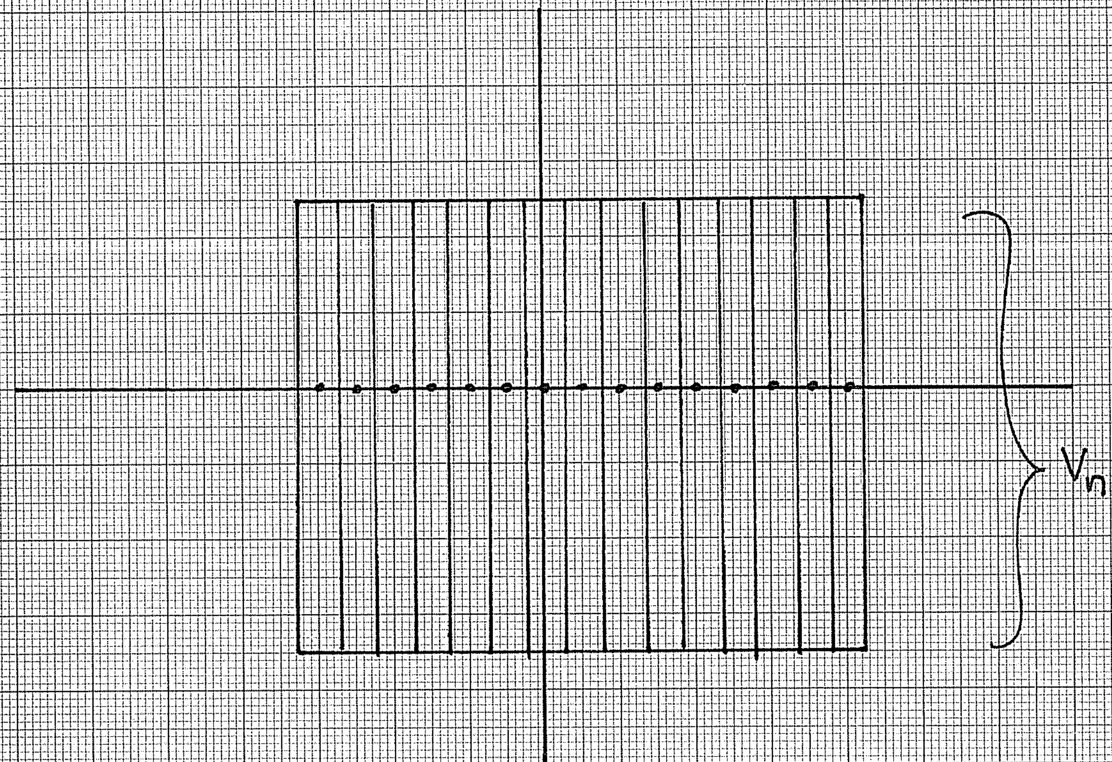


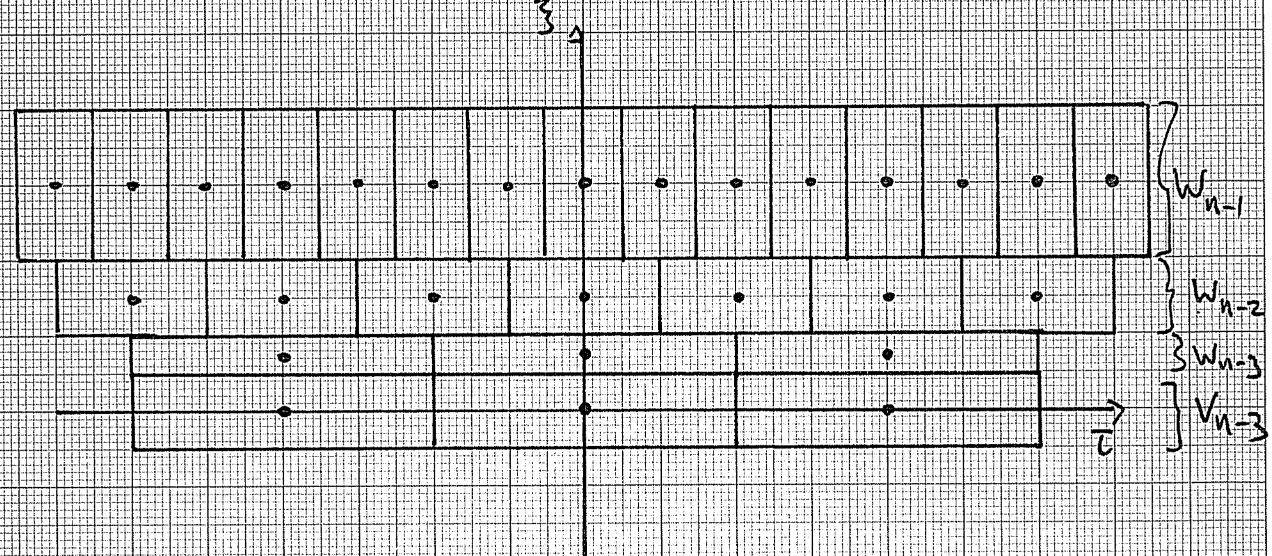


same here



Time - Frequency Picture





Shorthand Notation

$$s_j^n = \langle f, \varphi_j^n \rangle \qquad \vec{s}^{\,n} = (s_j^n)_{j \in \mathbb{Z}}$$

$$d_j^n = \langle f, \psi_j^n \rangle \qquad \vec{d}^{\,n} = (d_j^n)_{j \in \mathbb{Z}}$$

$$V_n = V_{n-L} \oplus W_{n-L} \oplus W_{n-L+1} \oplus \cdots \oplus W_{n-1}$$

$$f \in V_n = P_{n-L}f \oplus Q_{n-L}f \oplus Q_{n-L+1}f \oplus \cdots \oplus Q_{n-1}f$$

represented by

$$\vec{s}^{\,n} \qquad \vec{s}^{\,n-L} \qquad \vec{d}^{\,n-L} \qquad \vec{d}^{\,n-L+1} \qquad \cdots \qquad \vec{d}^{\,n-1}$$

Decomposition

compute $\vec{s}^{\,n-1}, \vec{d}^{\,n-1}$ from $\vec{s}^{\,n}$

Reconstruction

compute $\vec{s}^{\,n}$ from $\vec{s}^{\,n-1}, \vec{d}^{\,n-1}$

apply recursively

Calculating $\vec{s}^{\,n-1}$ from $\vec{s}^{\,n}$

$$f(x) = \sum_j s_j^n \varphi_j^n(x)$$

$$P_{n-1} f(x) = \sum_j s_k^{n-1} \varphi_k^{n-1}(x)$$

where
$$s_k^{n-1} = \langle f, \varphi_k^{n-1} \rangle = \langle \sum_j s_j^n \varphi_j^n, \varphi_k^{n-1} \rangle$$

$$= \sum_j \underbrace{\langle \varphi_j^n, \varphi_k^{n-1} \rangle}_{\text{need to compute this}} s_j^n$$

Fundamental Trick: $\quad V_{n-1} \subset V_n$

$\Rightarrow$ every function in V_{n-1} can be written as a linear combination of basis functions of V_n

In particular

$$\varphi_0^0(x) = \sum_i h_i \varphi_i^1(x) = \sum_i \langle \varphi_0^0, \varphi_i^1 \rangle \varphi_i^1(x)$$

$$\Leftrightarrow \quad \boxed{\varphi(x) = \sqrt{2} \sum_i h_i \varphi(2x - i)}$$

(Recursion Relation)

then
$$\varphi^{n-1}_k(x) = 2^{\frac{n-1}{2}} \varphi(2^{n-1}x - k)$$
$$= 2^{\frac{n-1}{2}} \left\{ \sqrt{2} \sum_i h_i \varphi(2^n x - 2k - i) \right\}$$
$$= 2^{\frac{n}{2}} \sum_i h_i \varphi(2^n x - (2k+i))$$
$$= \sum_i h_i \varphi^n_{2k+i}(x) = \sum_i h_{i-2k} \varphi^n_i(x)$$

---

so
$$\langle \varphi^n_j, \varphi^{n-1}_k \rangle = \langle \varphi^n_j, \sum_i h_{i-2k} \varphi^n_i \rangle$$
$$= \sum_i \underbrace{\langle \varphi^n_j, \varphi^n_i \rangle}_{\delta_{ij}} \overline{h}_{i-2k} = \overline{h}_{j-2k}$$

We get
$$\boxed{s^{n-1}_k = \sum_j \overline{h}_{j-2k} s^n_j}$$
convolution (filtering) followed by downsampling

likewise
$$\boxed{d^{n-1}_k = \sum_j \overline{g}_{j-2k} s^n_j}$$

where
$$\psi(x) = \sqrt{2} \sum_i g_i \varphi(2x - i)$$

<u>Reconstruction:</u>

$$\boxed{s^n_j = \sum_k h_{j-2k} s^{n-1}_k + \sum_k g_{j-2k} d^{n-1}_k}$$
upsampling followed by convolution

finite length vectors

$$(s_0^n \dots s_{N-1}^n) \qquad \text{length } N$$

method 1 extend vector by zeros

disadvantage: if h, g have length L
then $\vec{s}^{\,n-1}, \vec{d}^{\,n-1}$ have $\frac{N}{2} + \left(\frac{L}{2} - 1\right)$
nonzero coefficients each

method 2 periodize

disadvantage: edge effects

operation count

vector length N, filter length L

level 1: $2LN$ multiplications
level 2: $2L\left(\frac{N}{2}\right)$
$\vdots$
level k $2L\left(\frac{N}{2^{k-1}}\right)$

total: $2L\left(N + N/2 + N/4 + \dots\right) \approx 4LN = O(N)$

compare to FFT: $O(N \log N)$