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Near-optimal mean–variance controls under two-time-scale formulations and applications

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Although the mean–variance control was initially formulated for financial portfolio management problems in which one wants to maximize the expected return and control the risk, our motivations stem from highway vehicle platoon controls that aim to maximize highway utility while ensuring zero accident. This paper develops near-optimal mean–variance controls of switching diffusion systems. To reduce the computational complexity, with motivations from earlier work on singularly perturbed Markovian systems [Sethi and Zhang, *Hierarchical Decision Making in Stochastic Manufacturing Systems*, Birkhäuser, Boston, MA, 1994; Yin and Zhang, *Continuous-Time Markov Chains and Applications: A Singular Pertubation Approach*, Springer-Verlag, New York, 1998 and Yin et al., *Ann. Appl. Probab.* 10 (2000), pp. 549–572], we use a two-time-scale formulation to treat the underlying system, which is represented by the use of a small parameter. As the small parameter goes to 0, we obtain a limit problem. Using the limit problem as a guide, we construct controls for the original problem, and show that the control so constructed is nearly optimal.

Keywords: mean–variance control; regime-switching model; near-optimal control; two-time-scale; platoon application

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1. Introduction

This paper focuses on near-optimal controls of switching diffusions. Originating from the mean–variance portfolio optimization problems, our aim concentrates on reduction of computational complexity for switching diffusions where the discrete component (switching process) has a large state space. Decomposing the state space of the switching process into weakly connected subspaces and aggregating the states in each subspace into one state yield a limit system. Using the optimal controls of the limit system, we build controls for the original system, leading to near optimality. In addition to the traditional financial engineering applications, our motivation stems from formulations of platoon controls modelled by regime-switching systems involving two-time-scale Markov chains, which presents a twist of the mean–variance portfolio optimization. This paper is written in memory of our colleague and friend Michael Taksar, who had made significant contributions to stochastic control, financial mathematics and insurance risk theory; see [1] and numerous papers Michael published. The topic covered in this paper

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is related to what Michael had been working on. Meanwhile, the application in platoon control is a nice bifurcation from the usual finance applications.

Although the mean–variance control was initially formulated for financial portfolio management problems in which one wants to maximize the expected return and control the risk, our motivations also stem from highway vehicle platoon controls that aim to maximize highway utility while ensuring zero accident. As motivations, we identify three different but highly related aspects of platoon control problems that lead to different forms of the mean–variance type of problems that are investigated in this paper.

First, we consider the longitudinal inter-vehicle distance control. To increase highway utility, it is desirable to reduce the total length of a platoon, which intends to reduce inter-vehicle distances. This strategy, however, will increase the risk of collision in the presence of vehicle traffic uncertainties. A trade-off of these factors leads to a desired nominal length. Deviation from this nominal platoon length compromises either safety or highway utility. Since vehicle movements are subject to many random factors on road, weather and traffic conditions, the total platoon length is actually a stochastic process. The desire to control the platoon length towards its designated target (its mean) with small deviations (its variance) can be mathematically modelled as a mean–variance optimization problem with subsystem states as inter-vehicle distances.

The second scenario is communication resource allocation of bandwidths for vehicle-to-vehicle (V2V) communications. For a given maximum throughput of a platoon communication system, the communication system operator must find a way to assign this resource to different V2V channels. If the total bandwidth used is lower than assigned bandwidth, there will be resource waste. Conversely, usage of bandwidths over the budget may incur high costs or interfere with other platoons' operation. In this case, each channel's bandwidth usage is the state of the subsystem. Their summation is a random process and is desired to approach the maximum throughput (the desired mean at the terminal time) with small variations. Consequently, it becomes a mean–variance control problem.

Finally, we may view platoon fuel consumption (or similarly, total emission). The platoon fuel consumption is the summation of vehicle fuel consumptions. Due to variations in vehicle sizes and speeds, each vehicle's fuel consumption is a controlled random process. Trade-off between a platoon's team acceleration/manoeuvre capability and fuel consumption can be summarized in a desired platoon fuel consumption rate. Assigning fuels to different vehicles results in coordination of vehicle operations modelled by subsystem fuel rate dynamics. To control the platoon fuel consumption rate to be close to the designated value, one may formulate this as a mean–variance control problem.

Although the mean variance (MV) approach has never been applied to platoon controls, it has distinct advantages: (1) unlike heuristic methods such as neural network optimization and genetic algorithms, the MV method is simple but rigorous, (2) the MV method is computationally efficient, and (3) the form of the solution (i.e. efficient frontier) is readily applicable to assessing risks in platoon formation, hence is practically appealing.

The origin of the mean–variance optimization problem can be traced back to the Nobel-prize-winning work of Markowitz [5]. The salient feature of the model is that, in the context of finance, it enables an investor to seek desired expected return after specifying the acceptable risk level quantified by the variance of the return. The mean–variance approach has become the foundation of modern finance theory and has inspired numerous extensions and applications. Using the ideas of backward stochastic differential equations, the mean–variance problem for a continuous-time model was studied in [10] (see also

related work [3]). Note that in the mean–variance problems, the matrix related to the control (known as control weight) is not positive definite. To take into consideration the random environments not representable using the usual stochastic differential equation set-up, we developed more complex models with random switching in [11].

In this paper, we consider the case when the random process representing the discrete events (e.g., those in the environment) has a large state space. The physical system is such that not all of the discrete event states change at the same rate. Some of them vary rapidly and others change slowly. The fast and slow variations are in high contrast resulting in a two-timescale formulation. Taking advantage of the timescale separation, we use an averaging approach to analyse the system, which largely explores the weak and strong interactions of the switching diffusion due to the Markov chain. The rationale is to aggregate the states according to their jump rates and replace the actual system with its average. Using optimal control of the limit problem as a bridge, we then construct controls for the original system leading to feasible approximation schemes. In [4] and [9], we treated a class of linear quadratic problems with switching by concentrating on the associated Riccati systems of equations, whereas in this paper, we focus on mean–variance controls and examine certain associated systems of differential equations. We consider the case that the Markov chains have recurrent states as well as inclusion of transient states. These approximation schemes give us nearly optimal controls. Focusing on approximated optimality, we succeed in reducing the complexity of the underlying systems substantially.

The rest of the paper is arranged as follows. Section 2 begins with the formulation of the two-time-scale platoon problems. Section 3 proceeds with the study of the underlying mean–variance problem. Using completing square techniques, we derive the corresponding Riccati equations and optimal control for the non-definite control problem. Section 4 focuses on near-optimal controls of the mean–variance problems. First, Markov chains with recurrent states are treated and then inclusion of transient states is considered. Using probabilistic arguments and analytic techniques, the approximation schemes are shown to be nearly optimal. Finally, we conclude this paper with further thoughts and additional remarks in Section 5.

2. Formulation

We begin with a complete probability space $(\Omega, \mathcal{F}, P)$. Consider a time-homogeneous Markov chain in continuous time taking values in the state space $\mathcal{M} = \{1, 2, \ldots, m\}$ and a standard d-dimensional standard Brownian motion $w(t) = (w_1(t), w_2(t), \ldots, w_d(t))'$ (where a' denotes the transpose of $a \in \mathbb{R}^{l_1 \times l_2}$ with $l_i \geq 1$) that is independent of the Markov chain $\alpha(t)$. The Markov chain is used to represent discrete events and random environment. In [11], we considered the Markowitz's mean–variance portfolio selection problem in which the environment is allowed to vary randomly leading to a regime-switching model. In this paper, we continue using the set-up as in that of [11]. In addition to the finance applications, we have in mind the platoon control problems as mentioned in Section 1. Mathematically, the new feature considered here is that the state space of the discrete event process $\alpha(\cdot)$ is large. Treating mean–variance control problems thus requires handling of large-scale systems. Such a case naturally arises in the networked system formulation. The large-scale feature, however, renders the optimal control a difficult task. To reduce the computational complexity, we note that for the discrete event process (the Markov chain), not all states are varying at the same rate. Some clusters of states vary rapidly and others change slowly. Using the relative transition rates, we

decompose the state space $\mathcal{M}$ into subspaces $\mathcal{M}_i$ such that within each $\mathcal{M}_i$, the transitions happen frequently and among different clusters the transitions are relatively infrequently. To reflect the different transition rates, we let $\alpha(t) = \alpha^\varepsilon(t)$ where $\varepsilon > 0$ is a small parameter so that the generator of the Markov chain is given by

$$Q^\varepsilon = \frac{1}{\varepsilon}\tilde{Q} + \hat{Q}. \tag{2.1}$$

Define $\mathcal{F}_t = \sigma\{W(s), \alpha^\varepsilon(s) : 0 \le s \le t\}$. Denote

$$Q^\varepsilon f(\cdot)(i) = \sum_{j \neq i} q_{ij}^\varepsilon (f(\cdot, j) - f(\cdot, i)) \tag{2.2}$$

for a suitable $f(\cdot)$. Suppose that $x_i^\varepsilon(\cdot)$ are real-valued functions with $i = 0, \ldots, d_1$ such that

$$\mathrm{d}x_0^\varepsilon(t) = r(t, \alpha^\varepsilon(t))x_0^\varepsilon(t)\,\mathrm{d}t, \quad x_0^\varepsilon(0) = x_0, \quad \alpha^\varepsilon(0) = \alpha \tag{2.3}$$

for $\alpha^\varepsilon(t) \in \{1, 2, \ldots, m\}$. The flows of the other d_1 nodes follow geometric Brownian motion:

$$\begin{aligned} &\mathrm{d}x_i^\varepsilon(t) = x_i^\varepsilon(t)r_i(t, \alpha^\varepsilon(t))\,\mathrm{d}t + x_i^\varepsilon(t)\sigma_i(t, \alpha^\varepsilon(t))\,\mathrm{d}w(t), \quad x_i^\varepsilon(0) = x_i, \quad \alpha^\varepsilon(0) = \alpha \\ &\quad \text{for } i = 1, 2, \ldots, d_1, \ \ \alpha \in \mathcal{M}, \end{aligned} \tag{2.4}$$

where $\sigma_i(t, \alpha^\varepsilon(t)) = (\sigma_{i1}(t, \alpha^\varepsilon(t)), \sigma_{i2}(t, \alpha^\varepsilon(t)), \ldots, \sigma_{id}(t, \alpha^\varepsilon(t))) \in \mathbb{R}^{1\times d}$. In the finance application, $x_0^\varepsilon(\cdot)$ represents an investor's bank account value, whereas $x_i^\varepsilon(\cdot)$ for each $i = 1, \ldots, d_1$ is his wealth devoted to the ith stock. In the networked control problems, we use $x_i^\varepsilon(\cdot)$ to represent the flows of the ith node. We can represent the wealth of the investor or the total flows of the entire system as $x^\varepsilon(t)$.

For consistency with the current literature on the MV problems, we shall still use the term 'portfolio' in our network problems. In the traditional market analysis setting, a portfolio is a vector consisting of the dollar values of different stocks. When applied to our network systems, a portfolio will be the vector of inter-vehicle distances in platoon control, or individual channel throughput in communication resource allocation, or individual vehicle fuel consumption in platoon fuel management. The portfolio selection involves finding the strategy to select the proportion $n_i(\cdot)$ of the ith stock investment. Similarly, for the platoon problem, we need to decide the proportion $n_i(t)$ of the flow $x_i^\varepsilon(t)$ on node i. In these cases, we denote their sum as

$$x^\varepsilon(t) = \sum_{i=0}^{d_1} n_i(t)x_i^\varepsilon(t).$$

Then we have

$$\begin{aligned} &\mathrm{d}x^\varepsilon(t) = [r(t, \alpha^\varepsilon(t))x^\varepsilon(t) + B(t, \alpha^\varepsilon(t))u(t)]\,\mathrm{d}t + u'(t)\sigma(t, \alpha^\varepsilon(t))\,\mathrm{d}w(t), \\ &\quad t \in [0, T], \quad x^\varepsilon(0) = x = \sum_{i=0}^{d_1} n_i(0)x_i, \quad \alpha^\varepsilon(0) = \alpha, \end{aligned} \tag{2.5}$$

where

$$B(t,\alpha^{\varepsilon}(t)) = \big(r_1(t,\alpha^{\varepsilon}(t)) - r(t,\alpha^{\varepsilon}(t)), r_2(t,\alpha^{\varepsilon}(t)) - r(t,\alpha^{\varepsilon}(t)), \ldots, r_{d_1}(t,\alpha^{\varepsilon}(t)) - r(t,\alpha^{\varepsilon}(t))\big),$$

$\sigma(t,\alpha^{\varepsilon}(t)) = (\sigma_1(t,\alpha^{\varepsilon}(t)), \ldots, \sigma_{d_1}(t,\alpha^{\varepsilon}(t)))' \in \mathbb{R}^{d_1\times d}$ $u(t) = (u_1(t), \ldots, u_{d_1}(t))' \in \mathbb{R}^{d_1\times 1}$, and $u_i(t) = n_i(t)x_i(t)$ where $u_i(t)$ is the total amount of wealth in the ith stock or the amount of flow for node i at time t, for $i = 1, 2, \ldots, d_1$. We assume throughout this paper that $r(t,i)$, $B(t,i)$, $\sigma(t,i)$ are measurable and uniformly bounded in t and we also assume that the non-degeneracy condition is satisfied, i.e. there is some $\delta > 0$, $a(t,i) = \sigma(t,i)\sigma'(t,i) \geq \delta I$ for any $t \in [0,T]$ and each $i \in \mathcal{M}$. We denote by $L^2_{\mathcal{F}}(0,T;\mathbb{R}^{l_0})$ the set of all $\mathbb{R}^{l_0}$ valued, measurable stochastic processes $f(t)$ adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ such that $E\int_0^T |f(t)|^2\,\mathrm{d}t < \infty$.

Let $\mathcal{U}$, the set of control, be a compact subset in $\mathbb{R}^{d_1\times 1}$. The $u(\cdot)$ is said to be admissible if for a $\mathcal{U}$-valued control $u(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{d_1})$, Equation (2.5) has a unique solution $x^{\varepsilon}(\cdot)$ corresponding to $u(\cdot)$. In this case, we refer to $(x^{\varepsilon}(\cdot), u(\cdot))$ as an admissible pair. Our objective was to find an admissible control $u(\cdot)$ among all the admissible controls given that the expected terminal flow value of the whole system is $Ex^{\varepsilon}(T) = z$ for some given $z \in \mathbb{R}$, so that the risk measured by the variance of terminal of the flow is minimized. Thus, we have the following goal:

$$\min J(x,\alpha,u(\cdot)) = E[x^{\varepsilon}(T) - z]^2 \quad \text{subject to} \quad Ex^{\varepsilon}(T) = z. \tag{2.6}$$

Recall that the problem is called feasible if there is at least one portfolio satisfying all the constraints. The problem is called finite if it is feasible and the infimum of $J(x,\alpha,u(\cdot))$ is finite. An optimal portfolio to the above problem, if it ever exists, is called an efficient portfolio corresponding to z, and the corresponding $(Var\, x^{\varepsilon}(T), z) \in \mathbb{R}^2$ is called an efficient point. The set of all the efficient points is called the efficient frontier. The solution of the problem can be obtained by using the result of [11]. In fact, we can obtain the efficient frontier as well as the so-called mutual fund theorems. This, however, is not the end of the story but rather the starting point of this paper. In this paper, we consider the case that $|\mathcal{M}| = m$ is large, thus we have to solve a system of m equations where m is large. Computationally, this is rather cumbersome. Therefore, our effort is devoted to reducing the complexity.

3. Preliminary results

This section presents preliminary results concerning the solutions of systems. The results include feasibility, existence and uniqueness of the solution, and continuity. For the feasibility part of our problem, we present the following lemma. The detailed proof can be found in [11] (Theorem 3.3).

LEMMA 3.1. The mean–variance problem (2.6) is feasible for every $z \in \mathbb{R}$ if and only if

$$E\int_0^T |B(t,\alpha^{\varepsilon}(t))|^2\,\mathrm{d}t > 0. \tag{3.1}$$

Now let us proceed to the study of optimality. To handle the constraint part in problem (2.6), we apply the Lagrange multiplier technique and get unconstrained problem (see, e.g.

[11]) with multiplier $\lambda \in \mathbb{R}$:

$$\begin{aligned} &\min J(x, \alpha, u(\cdot), \lambda) = E[x^{\varepsilon}(T) + \lambda - z]^2 - \lambda^2 \\ &\text{subject to } \ Ex^{\varepsilon}(T) = z, \text{ with } (x^{\varepsilon}(\cdot),\ u(\cdot)) \text{ admissible.} \end{aligned} \tag{3.2}$$

To find the minimum of $J(x, \alpha, u(\cdot), \lambda)$, it suffices to choose $u(\cdot)$ so that $E(x^{\varepsilon}(T) + \lambda - z)^2$ is minimized. We regard this part as $J^{\varepsilon}(x, \alpha, u(\cdot))$ in what follows. In this section, we will proceed to solve the unconstrained problem (3.2). Let $v^{\varepsilon}(x, \alpha) = \inf_{u(\cdot)} J^{\varepsilon}(x, \alpha, u(\cdot))$ be the value function. First we define

$$\rho(t, i) = B(t, i)[\sigma(t, i)\sigma'(t, i)]^{-1}B'(t, i), \quad i \in \{1, 2, \ldots, m\}. \tag{3.3}$$

Consider the following two systems of ODEs for $i = 1, 2, \ldots, m$:

$$\dot{P}^{\varepsilon}(t, i) = P^{\varepsilon}(t, i)[\rho(t, i) - 2r(t, i)] - \sum_{j=1}^{m} q_{ij}^{\varepsilon} P^{\varepsilon}(t, j), \quad P^{\varepsilon}(T, i) = 1. \tag{3.4}$$

and

$$\begin{aligned} &\dot{H}^{\varepsilon}(t, i) = H^{\varepsilon}(t, i)r(t, i) - \frac{1}{P^{\varepsilon}(t, i)} \sum_{j=1}^{m} q_{ij}^{\varepsilon} P^{\varepsilon}(t, j) H^{\varepsilon}(t, j) + \frac{H^{\varepsilon}(t, i)}{P^{\varepsilon}(t, i)} \sum_{j=1}^{m} q_{ij}^{\varepsilon} P^{\varepsilon}(t, j), \\ &H^{\varepsilon}(T, i) = 1. \end{aligned} \tag{3.5}$$

The existence and uniqueness of solutions to the above two systems of equations are evident as both are linear with uniformly bounded coefficients. Applying the generalized Itô's formula to

$$v^{\varepsilon}(t, x^{\varepsilon}(t), i) = P^{\varepsilon}(t, i)(x^{\varepsilon}(t) + (\lambda - z)H^{\varepsilon}(t, i))^2$$

and using the completing square techniques, we obtain

$$\begin{aligned} &\mathrm{d}P^{\varepsilon}(t, i)\big[x^{\varepsilon}(t) + (\lambda - z)H^{\varepsilon}(t, i)\big]^2 \\ &\quad = 2P^{\varepsilon}(t, i)\big[x^{\varepsilon}(t) + (\lambda - z)H^{\varepsilon}(t, i)\big]\,\mathrm{d}x^{\varepsilon}(t) + P^{\varepsilon}(t, i)(\mathrm{d}x^{\varepsilon}(t))^2 \\ &\qquad + \sum_{j=1}^{m} q_{ij}^{\varepsilon} P^{\varepsilon}(t, j)\big[x^{\varepsilon}(t) + (\lambda - z)H^{\varepsilon}(t, j)\big]^2\,\mathrm{d}t \\ &\qquad + \dot{P}^{\varepsilon}(t, i)\big[x^{\varepsilon}(t) + (\lambda - z)H^{\varepsilon}(t, i)\big]^2\,\mathrm{d}t \\ &\qquad + 2P^{\varepsilon}(t, i)\big[x^{\varepsilon}(t) + (\lambda - z)H^{\varepsilon}(t, i)\big](\lambda - z)\dot{H}^{\varepsilon}(t, i)\,\mathrm{d}t. \end{aligned} \tag{3.6}$$

Therefore, by plugging in the dynamic equation satisfied by $P^\varepsilon(t,i)$ and $H^\varepsilon(t,i)$, we have the following expression:

$$\begin{aligned}
&\mathrm{d}P^\varepsilon(t,i)\big[x^\varepsilon(t)+(\lambda-z)H^\varepsilon(t,i)\big]^2\\
&\quad=P^\varepsilon(t,i)\big\{u'(t)\sigma(t,i)\sigma'(t,i)u(t)+2u'(t)B'(t,i)\big[x^\varepsilon(t)+(\lambda-z)H^\varepsilon(t,i)\big]\\
&\qquad+2r(t,i)x^\varepsilon(t)\big[x^\varepsilon(t)+(\lambda-z)H^\varepsilon(t,i)\big]\big\}\,\mathrm{d}t\\
&\qquad-\sum_{j=1}^m q_{ij}^\varepsilon P^\varepsilon(t,j)\big[x^\varepsilon(t)+(\lambda-z)H^\varepsilon(t,i)\big]^2\,\mathrm{d}t\\
&\qquad+2P^\varepsilon(t,i)\big[x^\varepsilon(t)+(\lambda-z)H^\varepsilon(t,i)\big](\lambda-z)\\
&\qquad\times\left\{H^\varepsilon(t,i)r(t,i)-\frac{1}{P^\varepsilon(t,i)}\sum_{j=1}^m q_{ij}^\varepsilon P^\varepsilon(t,j)H^\varepsilon(t,j)+\frac{H^\varepsilon(t,i)}{P^\varepsilon(t,i)}\sum_{j=1}^m q_{ij}^\varepsilon P^\varepsilon(t,j)\right\}\mathrm{d}t\\
&\qquad+\sum_{j=1}^m q_{ij}^\varepsilon P^\varepsilon(t,j)\big[x^\varepsilon(t)+(\lambda-z)H^\varepsilon(t,j)\big]^2\,\mathrm{d}t+[\rho(t,i)-2r(t,i)]P^\varepsilon(t,i)\\
&\qquad\times\big[x^\varepsilon(t)+(\lambda-z)H^\varepsilon(t,i)\big]^2\,\mathrm{d}t+(\cdots)\,\mathrm{d}w(t)\\
&\quad=P^\varepsilon(t,i)\Big\{\big(u(t)+(\sigma(t,i)\sigma'(t,i))^{-1}B'(t,i)\big[x^\varepsilon(t)+(\lambda-z)H^\varepsilon(t,i)\big]\big)'[\sigma(t,i)\sigma'(t,i)]\\
&\qquad\times\big(u(t)+(\sigma(t,i)\sigma'(t,i))^{-1}B'(t,i)\big[x^\varepsilon(t)+(\lambda-z)H^\varepsilon(t,i)\big]\big)\Big\}\,\mathrm{d}t\\
&\qquad+(\lambda-z)^2\sum_{j=1}^m q_{ij}^\varepsilon P^\varepsilon(t,j)\big[H^\varepsilon(t,j)-H^\varepsilon(t,i)\big]^2\,\mathrm{d}t+(\cdots)\,\mathrm{d}w(t).
\end{aligned}\tag{3.7}$$

Integrating both sides of the above equation from 0 to T and taking expectation, we obtain

$$\begin{aligned}
E[x^\varepsilon(T)+\lambda-z]^2&=P^\varepsilon(0,\alpha)\big[x+(\lambda-z)H^\varepsilon(0,\alpha)\big]^2+E\int_0^T(\lambda-z)^2\sum_{j=1}^m q_{ij}^\varepsilon P^\varepsilon(t,j)\\
&\qquad\times\big[H^\varepsilon(t,j)-H^\varepsilon(t,i)\big]^2\,\mathrm{d}t\\
&\quad+E\int_0^T P^\varepsilon(t,i)\big[u(t)-u^{\varepsilon,*}(t)\big]'[\sigma(t,i)\sigma'(t,i)]\big[u(t)-u^{\varepsilon,*}(t)\big]\,\mathrm{d}t.
\end{aligned}\tag{3.8}$$

Thus, the optimal control $u^{\varepsilon,*}$ has the form

$$\begin{aligned}
u^{\varepsilon,*}(t,\alpha^\varepsilon(t),x^\varepsilon(t))&=-\big(\sigma(t,\alpha^\varepsilon(t))\sigma'(t,\alpha^\varepsilon(t))\big)^{-1}B'(t,\alpha^\varepsilon(t))\\
&\qquad\times\big[x^\varepsilon(t)+(\lambda-z)H^\varepsilon(t,\alpha^\varepsilon(t))\big].
\end{aligned}\tag{3.9}$$

Now we introduce the following two lemmas here for the subsequent use.

LEMMA 3.2. The solution of Equations (3.4) and (3.5) satisfies $0 < P^\varepsilon(t,i) \le c$ and $0 < H^\varepsilon(t,i) \le 1$ for all $t \in [0,T]$, $i = 1, 2, \ldots, m$.

Proof. For the $H^\varepsilon(t,i)$, by employing the idea similar to [11] (Proposition 4.1), we can get the claim above. Here, we consider the case for $P^\varepsilon(t,i)$. First, by applying a variation of constant formula to (3.4) we have

$$P^\varepsilon(t,i) = \mathrm{e}^{-\int_t^T [\rho(s,i)-2r(s,i)-q_{ii}^\varepsilon]\,\mathrm{d}s} + \int_t^T \mathrm{e}^{-\int_t^s [\rho(\tau,i)-2r(\tau,i)-q_{ii}^\varepsilon]\,\mathrm{d}\tau} \sum_{j\neq i}^m q_{ij}^\varepsilon P^\varepsilon(s,j)\,\mathrm{d}s.$$

Construct a Picard sequence of $P_k^\varepsilon(\cdot\,,i)$ for $t \in [0,T]$, $i = 1, 2, \ldots, m$, $k = 0, 1, \ldots$ as follows:

$$P_0^\varepsilon(t,i) = 1, \quad P_{k+1}^\varepsilon(t,i) = \mathrm{e}^{-\int_t^T [\rho(s,i)-2r(s,i)-q_{ii}^\varepsilon]\,\mathrm{d}s} + \int_t^T \mathrm{e}^{-\int_t^s [\rho(\tau,i)-2r(\tau,i)-q_{ii}^\varepsilon]\,\mathrm{d}\tau} \sum_{j\neq i}^m q_{ij}^\varepsilon P_k^\varepsilon(s,j)\,\mathrm{d}s.$$

Noting that $q_{ij}^\varepsilon \ge 0$ for all $j \neq i$, we have for $k = 0, 1, \ldots$

$$P_k^\varepsilon(t,i) \ge \mathrm{e}^{-\int_t^T [\rho(s,i)-2r(s,i)-q_{ii}^\varepsilon]\,\mathrm{d}s} > 0.$$

Realizing that $P^\varepsilon(t,i)$ is the limit of the Picard sequence $P_k^\varepsilon(t,i)$ as $k \to \infty$, thus $P^\varepsilon(t,i) > 0$. To get the upper bound, we first consider the bounds of value function $v^\varepsilon(x,\alpha)$. Clearly, $v^\varepsilon(x,\alpha) \ge 0$ since $J^\varepsilon(x,\alpha,u(\cdot)) \ge 0$ for all admissible $u(\cdot)$. We choose $u_0(t) = -ax^\varepsilon(t)$, a is a non-zero vector in $\mathbb{R}^{d_1}$ and $x^\varepsilon(t) = \tilde{x}$, then we have $E(x^\varepsilon(T))^2 \le \tilde{x}^2 + k\int_t^T E(x^\varepsilon(s))^2\,\mathrm{d}s$ according to Itô's formula. We further have $E(x^\varepsilon(T))^2 \le \tilde{x}^2\,\mathrm{e}^{kT}$ by virtue of Gronwall's inequality for all $t \in [0,T]$. Now, note that for $0 \le t \le T$,

$$v^\varepsilon(\tilde{x},i) \le J^\varepsilon(\tilde{x},i,u(\cdot)) \le E[x^\varepsilon(T) + \lambda - z]^2 \le 2\tilde{x}^2\,\mathrm{e}^{kT} + 2(\lambda - z)^2.$$

Then we have

$$P^\varepsilon(t,i)(\tilde{x} + (\lambda - z)H^\varepsilon(t,i))^2 \le 2\tilde{x}^2\,\mathrm{e}^{kT} + 2(\lambda - z)^2.$$

Dividing both sides of this inequality by $\tilde{x}^2$ and setting $\tilde{x} \to \infty$, we have $P^\varepsilon(t,i) \le 2\,\mathrm{e}^{kT}$. □

LEMMA 3.3. For $i \in \mathcal{M}$, the solutions of (3.4) and (3.5) are uniformly Lipschitz on $[0,T]$.

Proof. Let us just consider the part of $P^\varepsilon(t,i)$ since the proof for the case of $H^\varepsilon(t,i)$ is similar. Given that the solution for Equation (3.4) is

$$\begin{aligned}P^\varepsilon(t,i) &= \mathrm{e}^{-\int_t^T[\rho(s,i)-2r(s,i)]\,\mathrm{d}s} + \int_t^T \mathrm{e}^{-\int_t^s[\rho(\tau,i)-2r(\tau,i)]\,\mathrm{d}\tau}\sum_{j=1}^m q_{ij}^\varepsilon P^\varepsilon(s,j)\,\mathrm{d}s\\ &= \mathrm{e}^{-\int_t^{t+\Delta}[\rho(s,i)-2r(s,i)]\,\mathrm{d}s-\int_{t+\Delta}^T[\rho(s,i)-2r(s,i)]\,\mathrm{d}s}\\ &\quad + \int_t^{t+\Delta} \mathrm{e}^{-\int_t^s[\rho(\tau,i)-2r(\tau,i)]\,\mathrm{d}\tau}\sum_{j=1}^m q_{ij}^\varepsilon P^\varepsilon(s,j)\,\mathrm{d}s\\ &\quad + \int_{t+\Delta}^T \mathrm{e}^{-\int_t^{t+\Delta}[\rho(\tau,i)-2r(\tau,i)]\,\mathrm{d}\tau-\int_{t+\Delta}^s[\rho(\tau,i)-2r(\tau,i)]\,\mathrm{d}\tau}\sum_{j=1}^m q_{ij}^\varepsilon P^\varepsilon(s,j)\,\mathrm{d}s.\end{aligned}$$

Given that

$$P^\varepsilon(t+\Delta,i) = \mathrm{e}^{-\int_{t+\Delta}^T[\rho(s,i)-2r(s,i)]\,\mathrm{d}s} + \int_{t+\Delta}^T \mathrm{e}^{-\int_{t+\Delta}^s[\rho(\tau,i)-2r(\tau,i)]\,\mathrm{d}\tau}\sum_{j=1}^m q_{ij}^\varepsilon P^\varepsilon(s,j)\,\mathrm{d}s.$$

Then we have

$$\begin{aligned}|P^\varepsilon(t,i) - P^\varepsilon(t+\Delta,i)| &= \left|\mathrm{e}^{-\int_{t+\Delta}^T[\rho(s,i)-2r(s,i)]\,\mathrm{d}s}\left(1-\mathrm{e}^{-\int_t^{t+\Delta}[\rho(s,i)-2r(s,i)]\,\mathrm{d}s}\right)\right.\\ &\quad + \int_{t+\Delta}^T \mathrm{e}^{-\int_{t+\Delta}^s[\rho(\tau,i)-2r(\tau,i)]\,\mathrm{d}\tau}\left(1-\mathrm{e}^{-\int_t^{t+\Delta}[\rho(\tau,i)-2r(\tau,i)]\,\mathrm{d}\tau}\right)\sum_{j=1}^m q_{ij}^\varepsilon P^\varepsilon(s,j)\,\mathrm{d}s\\ &\quad \left.+\int_t^{t+\Delta} \mathrm{e}^{-\int_t^s[\rho(\tau,i)-2r(\tau,i)]\,\mathrm{d}\tau}\sum_{j=1}^m q_{ij}^\varepsilon P^\varepsilon(s,j)\,\mathrm{d}s\right|.\end{aligned}$$

As $\Delta \to 0$,

$$1-\mathrm{e}^{-\int_t^{t+\Delta}[\rho(\tau,i)-2r(\tau,i)]\,\mathrm{d}\tau} \to 0$$

and

$$\int_t^{t+\Delta} \mathrm{e}^{-\int_t^s[\rho(\tau,i)-2r(\tau,i)]\,\mathrm{d}\tau}\sum_{j=1}^m q_{ij}^\varepsilon P^\varepsilon(s,j)\,\mathrm{d}s \to 0$$

hold for any $t \in [0,T]$ uniformly, therefore, $P^\varepsilon(t,i)$ is uniformly Lipschitz on $[0,T]$. □

Due to the large dimensionality, it is highly computation intensive to obtain the optimal controls. To overcome the difficulty, we design a near-optimal control scheme (see related work [6]). We will show that as $\varepsilon \to 0$, there is a limit problem. For the limit problem, we can obtain optimal controls as given in [11]. Then we use the optimal control of the limit problem to construct controls of the original problem and show that the constructed control is asymptotically optimal.

4. Near-optimal controls

4.1 Recurrent states

Assume that $\tilde{Q}$ can be put into a block-diagonal form $\tilde{Q} = \operatorname{diag}(\tilde{Q}^1, \ldots, \tilde{Q}^l)$ in which $\tilde{Q}^k \in \mathbb{R}^{m_k \times m_k}$ are irreducible for $k = 1, 2, \ldots, l$ and $\sum_{k=1}^{l} m_k = m$. $\tilde{Q}^k$ denotes the kth block matrix in $\tilde{Q}$. Denote by $\mathcal{M}_k = \{s_{k1}, s_{k2}, \ldots, s_{km_k}\}$ the states corresponding to $\tilde{Q}^k$ and note

$$\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \cdots \cup \mathcal{M}_l.$$

Note that the $\tilde{Q}^k = (\tilde{q}^k_{ij})_{m_k \times m_k}$ and $\hat{Q} = (\hat{q}_{ij})_{m \times m}$ are generators.

The slow and fast components are coupled through weak and strong interactions in the sense that the underlying Markov chain fluctuates rapidly within a single-group $\mathcal{M}_k$ and jumps less frequently among groups $\mathcal{M}_k$ and $\mathcal{M}_j$ for $k \neq j$. By aggregating the states in $\mathcal{M}_k$ as one state k, we obtain an aggregated process $\bar{\alpha}^\varepsilon(\cdot)$ defined by $\bar{\alpha}^\varepsilon(t) = k$ when $\alpha^\varepsilon(t) \in \mathcal{M}_k$. Although $\bar{\alpha}^\varepsilon(t)$ is generally not Markovian, by virtue of [7] (Theorem7.4), $\bar{\alpha}^\varepsilon(\cdot)$ converges weakly to a Markov chain $\bar{\alpha}(\cdot)$ with generator $\bar{Q} = (\bar{q}_{ij})$ satisfying

$$\bar{Q} = \operatorname{diag}(\mu^1, \mu^2, \ldots, \mu^l)\hat{Q}\operatorname{diag}(\mathbb{1}_{m_1}, \mathbb{1}_{m_2}, \ldots, \mathbb{1}_{m_l}),$$

where μ^k is the stationary distribution associated with $\tilde{Q}^k$, $k = 1, 2, \ldots, l$, and $\mathbb{1}_n = (1, 1, \ldots, 1) \in \mathbb{R}^{n \times 1}$. For subsequent use, we define $\bar{F}(t, k) = \sum_{j=1}^{m_k} \mu^k_j F(t, s_{kj})$ for $F(t, s_{kj}) = r(t, s_{kj}), B(t, s_{kj})$ and $\rho(t, s_{kj})$. The following theorems are concerned with the convergence and nearly optimal control.

THEOREM 4.1. For $k = 1, 2, \ldots, l$ and $j = 1, 2, \ldots, m_k$, $P^\varepsilon(t, s_{kj}) \to \bar{P}(t, k)$ and $H^\varepsilon(t, s_{kj}) \to \bar{H}(t, k)$ uniformly on $[0, T]$ as $\varepsilon \to 0$, where $\bar{P}(t, k)$ and $\bar{H}(t, k)$ are the unique solutions of the following differential equations for $k = 1, 2, \ldots, l$,

$$\dot{\bar{P}}(t, k) = (\bar{\rho}(t, k) - 2\bar{r}(t, k))\bar{P}(t, k) - \bar{Q}\bar{P}(t, \cdot)(k), \quad \bar{P}(T, k) = 1. \tag{4.1}$$

and

$$\begin{aligned} \dot{\bar{H}}(t, k) &= \bar{r}(t, k)\bar{H}(t, k) - \frac{1}{\bar{P}(t, k)}\bar{Q}\bar{P}(t, \cdot)\bar{H}(t, \cdot)(k) + \frac{\bar{H}(t, k)}{\bar{P}(t, k)}\bar{Q}\bar{P}(t, \cdot)(k), \\ \bar{H}(T, k) &= 1. \end{aligned} \tag{4.2}$$

Proof. We prove the convergence of P^ε (the proof of H^ε is similar). By virtue of Lemmas 3.2 and 3.3, $P^\varepsilon(t, s_{kj})$ is equicontinuous and uniformly bounded, it follows from Arzela–Ascoli theorem that, for each sequence of $\varepsilon \to 0$, a further subsequence exists (we still use the index ε for the sake of simplicity) such that $P^\varepsilon(t, s_{kj})$ converges uniformly on $[0, T]$ to a continuous function, say, $P^0(t, s_{kj})$. First, we show $P^0(t, s_{kj})$ is independent of j. Given that

$$P^\varepsilon(t, s_{kj}) = 1 - \int_t^T \left[P^\varepsilon(s, s_{kj})\big(\rho(s, s_{kj}) - 2r(s, s_{kj})\big) - Q^\varepsilon P^\varepsilon(s, \cdot)(s_{kj})\right] \mathrm{d}s.$$

Multiplying both sides of above equation by ε yields

$$0 = \lim_{\varepsilon\to 0}\int_t^T \tilde{Q}^k P^\varepsilon(s,\cdot)(s_{kj})\,\mathrm{d}s = \int_t^T \tilde{Q}^k P^0(s,\cdot)(s_{kj})\,\mathrm{d}s.$$

Thus, in view of the continuity of $P^0(t,\cdot)(s_{kj})$, we obtain

$$\tilde{Q}^k P^0(t,\cdot)(s_{kj}) = 0 \quad \text{for } t\in[0,T]. \tag{4.3}$$

Given the fact that $\tilde{Q}^k$ is irreducible, we have $P^0(t,s_{kj}) = P^0(t,k)$ which is independent of j. Now let us multiply $P^\varepsilon(t,s_{kj})$ by μ_j^k and then add the index j. Recall the definition of $\bar{F}(t,k)$, we have the following equation:

$$\sum_{j=1}^{m_k}\mu_j^k P^\varepsilon(t,s_{kj}) = 1 - \sum_{j=1}^{m_k}\mu_j^k\int_t^T\left[P^\varepsilon(s,s_{kj})(\rho(s,s_{kj}) - 2r(s,s_{kj})) - Q^\varepsilon P^\varepsilon(s,\cdot)(s_{kj})\right]\mathrm{d}s.$$

Letting $\varepsilon\to 0$ and noting that uniform convergence of $P^\varepsilon(t,s_{kj})\to P^0(t,k)$ and μ^k is the stationary distribution corresponding to $\tilde{Q}^k$, we have

$$\left(\sum_{j=1}^{m_k}\mu_j^k\hat{Q}\mathbb{1}_{m_k}\right)P^0(t,\cdot)(k) = \bar{Q}P^0(t,\cdot)(k).$$

Therefore, we obtain

$$P^0(t,k) = 1 - \int_t^T\left(P^0(s,k)(\bar{\rho}(s,k) - 2\bar{r}(s,k) - \bar{Q}P^0(s,\cdot)(k)\right)\mathrm{d}s.$$

Then the uniqueness of solution of the Riccati equation implies $P^0(s,k) = \bar{P}(s,k)$. Therefore, $P^\varepsilon(t,s_{kj})\to\bar{P}(t,k)$ and the proof is thus concluded. □

It follows that $P^\varepsilon(t,s_{kj})\to\bar{P}(t,k)$ and $H^\varepsilon(t,s_{kj})\to\bar{H}(t,k)$. We thus have $v^\varepsilon(t,s_{kj},x)\to\bar{v}(t,k,x)$ as $\varepsilon\to 0$, in which $\bar{v}(t,k,x) = \bar{P}(t,k)(x+(\lambda-z)\bar{H}(t,k))^2$, where $\bar{v}(t,k,x)$ corresponds to the value function of a limit problem. Let $\mathcal{U}$ denote the control set for the limit problem: $\mathcal{U} = \{U = (U^1,U^2,\ldots,U^l) : U^k = (u^{k1},u^{k2},\ldots,u^{km_k}), u^{kj}\in\mathbb{R}^{d_1}\}$. Define

$$f(t,x,k,U) = \sum_{j=1}^{m_k}\mu_j^k r(t,s_{kj})x + \sum_{j=1}^{m_k}\mu_j^k B(t,s_{kj})u^{kj}(t)$$

and

$$g(t,k,U) = (g_1(t,k,U),\ldots,g_d(t,k,U)) \quad \text{with}$$

$$g_i(t,k,U) = \sqrt{\sum_{j=1}^{m_k}\mu_j^k\left(\sum_{n=1}^{d_1}u_n^{kj}\sigma_{ni}(t,\alpha^\varepsilon(t))\right)^2}.$$

Recall that $\sigma(t,\alpha^\varepsilon(t)) = (\sigma_{ni}(t,\alpha^\varepsilon(t))) \in \mathbb{R}^{d_1\times d}$ and note that u_n^{kj} is the nth component of the d_1-dimensional variable. The corresponding dynamic system of the state is

$$\mathrm{d}x(t) = f(t,x(t),\bar\alpha(t),U(t))\,\mathrm{d}t + \sum_{i=1}^{d} g_i(t,\bar\alpha(t),U(t))\,\mathrm{d}w_i(t). \tag{4.4}$$

where $\bar\alpha(\cdot)\in\{1,2,\ldots,l\}$ is a Markov chain generated by $\bar Q$ with $\bar\alpha(0)=\alpha$. Calculation similar to (3.6) and (3.7) shows that the optimal control for this limit problem is

$$\begin{aligned} U^*(t) &= \big(U^{1*}(t,x), U^{2*}(t,x), \ldots, U^{l*}(t,x)\big), \quad \text{with}\\ U^{k*}(t,x) &= \big(u^{k1*}(t,x), u^{k2*}(t,x), \ldots, u^{km_k*}(t,x)\big),\\ u^{kj*}(t,x) &= -\big(\sigma(t,s_{kj})\sigma'(t,s_{kj})\big)^{-1}B'(t,s_{kj})[x+(\lambda-z)\bar H(t,k)]. \end{aligned}$$

In the following, we denote nth component of the optimal control for this limit system as $u_n^{kj*}(t,x)$. Using such controls, we construct

$$u^\varepsilon(t,\alpha^\varepsilon(t),x) = \sum_{k=1}^{l}\sum_{j=1}^{m_k} I_{\{\alpha^\varepsilon(t)=s_{kj}\}}u^{kj*}(t,x) \tag{4.5}$$

for the original problem. This control can also be written as if $\alpha^\varepsilon(t)\in\mathcal{M}_k$, $u^\varepsilon(t,\alpha^\varepsilon(t),x) = -(\sigma(t,\alpha^\varepsilon(t))\sigma'(t,\alpha^\varepsilon(t)))^{-1}B'(t,\alpha^\varepsilon(t))[x+(\lambda-z)\bar H(t,\bar\alpha^\varepsilon(t))]$. To proceed, we present the following lemmas first.

LEMMA 4.2. For a positive T and any $k=1,2,\ldots,l$, $j=1,2,\ldots,m_k$,

$$\sup_{0\le t\le T} E\left|\int_0^t \Big[I_{\{\alpha^\varepsilon(s)=s_{kj}\}} - \mu_j^k I_{\{\bar\alpha^\varepsilon(s)=k\}}\Big]x^\varepsilon(s)r(s,s_{kj})\,\mathrm{d}s\right|^2 \to 0 \quad \text{as } \varepsilon\to 0. \tag{4.6}$$

The proof is omitted for brevity.

LEMMA 4.3. For any $k=1,2,\ldots,l$, $j=1,2,\ldots,m_k$,

$$E\big(I_{\{\bar\alpha^\varepsilon(s)=k\}} - I_{\{\bar\alpha(s)=k\}}\big)^2 \to 0 \quad \text{as } \varepsilon\to 0. \tag{4.7}$$

Proof. Similar to [7] (Theorem 7.30), we can show that $(I_{\{\bar\alpha^\varepsilon(\cdot)=1\}},\ldots,I_{\{\bar\alpha^\varepsilon(\cdot)=l\}})$ converges weakly to $(I_{\{\bar\alpha(\cdot)=1\}},\ldots,I_{\{\bar\alpha(\cdot)=l\}})$ in $(D[0,T]:\mathbb{R}^l)$ as $\varepsilon\to 0$. By means of Cramér-Word's device, for each $i\in\mathcal{M}$, $I_{\{\bar\alpha^\varepsilon(\cdot)=i\}}$ converges weakly to $I_{\{\bar\alpha(\cdot)=i\}}$. Then by virtue of the Skorohod representation (with a slight abuse of notation), we may assume $I_{\{\bar\alpha^\varepsilon(\cdot)=i\}}\to I_{\{\bar\alpha(\cdot)=i\}}$ w.p.1. without change of notation. Now by dominance convergence theorem, we can conclude the proof. □

THEOREM 4.4. The control $u^\varepsilon(t)$ defined in (4.5) is nearly optimal in that $\lim_{\varepsilon\to 0}|J^\varepsilon(\alpha,x,u^\varepsilon(\cdot)) - v^\varepsilon(\alpha,x)| = 0$.

Proof. Recall the definition of $\rho(t, s_{kj})$ in (3.3) and note that the constructed control is given as $u^\varepsilon(t, x, \alpha^\varepsilon(t)) = -(\sigma(t, \alpha^\varepsilon(t))\sigma'(t, \alpha^\varepsilon(t)))^{-1}B'(t, \alpha^\varepsilon(t))\ [x + (\lambda - z)\bar{H}(t, \bar{\alpha}^\varepsilon(t))]$. Then $x^\varepsilon(t)$ follows:

$$\mathrm{d}x^\varepsilon(t) = \sum_{k=1}^{l}\sum_{j=1}^{m_k}\left[r(t, s_{kj})x^\varepsilon(t) - \rho(t, s_{kj})x^\varepsilon(t) - \rho(t, s_{kj})(\lambda - z)\bar{H}(t, k)\right]I_{\{\alpha^\varepsilon(t)=s_{kj}\}}\,\mathrm{d}t$$

$$+\sum_{i=1}^{d}\sqrt{\sum_{k=1}^{l}\sum_{j=1}^{m_k}\left(\sum_{n=1}^{d_1}u_n^\varepsilon(t, x^\varepsilon(t), \alpha^\varepsilon(t))\sigma_{ni}(t, \alpha^\varepsilon(t))\right)^2 I_{\{\alpha^\varepsilon(t)=s_{kj}\}}}\,\mathrm{d}w_i(t), \quad x^\varepsilon(0) = \hat{x}.$$

The cost function $J^\varepsilon(\alpha, x, u^\varepsilon(\cdot)) = E[x^\varepsilon(T) + \lambda - z]^2$. Let $x^*(t)$ be the optimal trajectory of the limit problem. Recall the definition of $f(\cdot)$ and $g(\cdot)$ in Theorem 4.1. Then

$$\mathrm{d}x^*(t) = f(t, x^*(t), \bar{\alpha}(t), U^*(t))\,\mathrm{d}t + \sum_{i=1}^{d} g_i(t, \bar{\alpha}(t), U^*(t))\,\mathrm{d}w_i(t), \quad x^*(0) = \hat{x}.$$

Similar to the methods in [7] (Theorem 9.8), for all $\alpha \in \mathcal{M}_k$, and $k = 1, 2, \ldots, l$,

$$\lim_{\varepsilon\to 0} v^\varepsilon(x, \alpha) = \bar{v}(x, k).$$

Here $\bar{v}(x, k)$ is the value function of the limit problem. For any $\alpha \in \mathcal{M}_k, k = 1, 2, \ldots, l$,

$$0 \le |J^\varepsilon(x, u^\varepsilon(\cdot), \alpha) - v^\varepsilon(x, \alpha)| = |J^\varepsilon(x, u^\varepsilon(\cdot), \alpha) - \bar{v}(x, k) + \bar{v}(x, k) - v^\varepsilon(x, \alpha)|.$$

To establish the assertion, it suffices to show that

$$|J^\varepsilon(x, u^\varepsilon(\cdot), \alpha) - \bar{v}(x, k)| \to 0,$$

$$\begin{aligned}
|J^\varepsilon(x, u^\varepsilon(\cdot), \alpha) - \bar{v}(x, \alpha)| &= |E[x^\varepsilon(T) + \lambda - z]^2 - E[x^*(T) + \lambda - z]^2| \\
&= |Ex^\varepsilon(T)^2 + 2(\lambda - z)Ex^\varepsilon(T) - Ex^*(T) - 2(\lambda - z)Ex^*(T)| \\
&\le CE^{1/2}[x^\varepsilon(T) - x^*(T)]^2
\end{aligned} \tag{4.8}$$

for some constant C. Here, Hölder inequality and finite second moment of $x^\varepsilon(T)$ and $x^*(T)$ are used. Note that we can write $E(x^\varepsilon(T) - x^*(T))^2$ as follows:

$$\begin{aligned}
&E(x^\varepsilon(T) - x^*(T))^2 \\
&\quad\le K\sum_{k=1}^{l}\sum_{j=1}^{m_k}E\left(\int_0^T\left[r(s, s_{kj})x^\varepsilon(s)\left(I_{\{\alpha^\varepsilon(s)=s_{kj}\}} - \mu_j^k I_{\{\bar{\alpha}^\varepsilon(s)=k\}}\right)\right]\mathrm{d}s\right)^2 \\
&\qquad + K\sum_{k=1}^{l}\sum_{j=1}^{m_k}E\left(\int_0^T\left[\mu_j^k r(s, s_{kj})(x^\varepsilon(s) - x^*(s))I_{\{\bar{\alpha}^\varepsilon(s)=k\}}\right]\mathrm{d}s\right)^2 \\
&\qquad + K\sum_{k=1}^{l}\sum_{j=1}^{m_k}E\left(\int_0^T\left[\mu_j^k r(s, s_{kj})x^*(s)\left(I_{\{\bar{\alpha}^\varepsilon(s)=k\}} - I_{\{\bar{\alpha}(s)=k\}}\right)\right]\mathrm{d}s\right)^2
\end{aligned} \tag{4.9}$$

$$
\begin{aligned}
&-K\sum_{k=1}^{l}\sum_{j=1}^{m_k}E\left(\int_0^T\rho(s,s_{kj})x^{\varepsilon}(s)\left(I_{\{\alpha^{\varepsilon}(s)=s_{kj}\}}-\mu_j^kI_{\{\bar{\alpha}^{\varepsilon}(s)=k\}}\right)\mathrm{d}s\right)^2\\
&+K\sum_{k=1}^{l}\sum_{j=1}^{m_k}E\left(\int_0^T\left[\mu_j^k\rho(s,s_{kj})(x^{\varepsilon}(s)-x^*(s))I_{\{\bar{\alpha}^{\varepsilon}(s)=k\}}\right]\mathrm{d}s\right)^2\\
&+K\sum_{k=1}^{l}\sum_{j=1}^{m_k}E\left(\int_0^T\mu_j^k\rho(s,s_{kj})x^*(s)\left(I_{\{\bar{\alpha}^{\varepsilon}(s)=k\}}-I_{\{\bar{\alpha}(s)=k\}}\right)\mathrm{d}s\right)^2 \qquad (4.9)\\
&-K\sum_{k=1}^{l}\sum_{j=1}^{m_k}E\left(\int_0^T\left[\rho(s,s_{kj})(\lambda-z)\bar{H}(s,k)\left(I_{\{\alpha^{\varepsilon}(s)=s_{kj}\}}-\mu_j^kI_{\{\bar{\alpha}^{\varepsilon}(s)=k\}}\right)\right]\mathrm{d}s\right)^2\\
&+K\sum_{k=1}^{l}\sum_{j=1}^{m_k}E\left(\int_0^T\left[\rho(s,s_{kj})(\lambda-z)\bar{H}(s,k)\mu_j^k\left(I_{\{\bar{\alpha}^{\varepsilon}(s)=k\}}-I_{\{\bar{\alpha}(s)=k\}}\right)\right]\mathrm{d}s\right)^2+D,
\end{aligned}
$$

where

$$
\begin{aligned}
D=KE\Bigg[\int_0^T\sum_{i=1}^{d}\Bigg[&\sqrt{\sum_{k=1}^{l}\sum_{j=1}^{m_k}\Big(\sum_{n=1}^{d_1}u_n^{\varepsilon}(s,x^{\varepsilon}(s),\alpha^{\varepsilon}(s))\sigma_{ni}(s,\alpha^{\varepsilon}(s))\Big)^2I_{\{\alpha^{\varepsilon}(s)=s_{kj}\}}}\\
&-\sqrt{\sum_{k=1}^{l}\sum_{j=1}^{m_k}\mu_j^k\left(\sum_{n=1}^{d_1}u_n^{kj*}(s,x^*(s),\bar{\alpha}(s))\sigma_{ni}(s,\alpha^{\varepsilon}(s))\right)^2I_{\{\bar{\alpha}(s)=k\}}}\Bigg]\mathrm{d}w_i(s)\Bigg]^2.
\end{aligned}
$$

First, we use Lemma 4.2, Lemma 4.3 and Hölder inequality repeatedly to handle the drift part. For the diffusion part, we realize that

$$
\begin{aligned}
D\le KE\int_0^T\sum_{i=1}^{d}\Bigg[&\sqrt{\sum_{k=1}^{l}\sum_{j=1}^{m_k}\left(\sum_{n=1}^{d_1}u_n^{\varepsilon}(s,x^{\varepsilon}(s),\alpha^{\varepsilon}(s))\sigma_{ni}(s,\alpha^{\varepsilon}(s))\right)^2\left[I_{\{\alpha^{\varepsilon}(s)=s_{kj}\}}-\mu_j^kI_{\{\bar{\alpha}^{\varepsilon}(s)=k\}}\right]}\\
&+\sqrt{\sum_{k=1}^{l}\sum_{j=1}^{m_k}\mu_j^k\left(\sum_{n=1}^{d_1}u_n^{kj*}(s,x^*(s),\bar{\alpha}(s))\sigma_{ni}(s,\alpha^{\varepsilon}(s))\right)^2\left[I_{\{\bar{\alpha}^{\varepsilon}(s)=k\}}-I_{\{\bar{\alpha}(s)=k\}}\right]}\\
&+\left(x^{\varepsilon}(s)-x^*(s)\right)\Bigg]^2\mathrm{d}s.
\end{aligned}
$$

Here, we plugged in the control constructed in (4.5) for the last term above and utilized the non-degeneracy assumption mentioned in the previous section. Then we can use property of stochastic integral, dominance convergence theorem, similar techniques involved in dealing with the drift part and the finite second moment of $x^{\varepsilon}(\cdot)$ and $x^*(\cdot)$ to proceed with the diffusion part. Finally, after detailed calculation, we have $E(x^{\varepsilon}(T)-x^*(T))^2\le o(\varepsilon)+K\int_0^TE(x^{\varepsilon}(s)-x^*(s))^2\,\mathrm{d}s$. Now with the help of Gronwall's inequality, we obtain $E(x^{\varepsilon}(T)-x^*(T))^2\to 0$ as $\varepsilon\to 0$. The proof is thus concluded. □

Remark 4.5. Note that efficient frontier, efficient portfolio and minimum variance for the limit system can be obtained similar to [11] (Theorems 5.1–5.3). We provide the discussion below and omit the detailed proofs. The discussions below also carry over to the case in which the Markov chain has transient states to be studied in the next section.

(1) If (3.1) holds, we have

$$\bar{P}(0,\alpha)\bar{H}(0,\alpha)^2 + \theta - 1 < 0. \tag{4.10}$$

and the efficient control corresponding to z is

$$u^{kj*}(t,x) = -\big(\sigma(t,s_{kj})\sigma'(t,s_{kj})\big)^{-1}B'(t,s_{kj})\big[x + (\lambda^* - z)\bar{H}(t,k)\big] \tag{4.11}$$

in which

$$\lambda^* - z = \frac{z - \bar{P}(0,\alpha)\bar{H}(0,\alpha)\hat{x}}{\bar{P}(0,\alpha)\bar{H}(0,\alpha)^2 + \theta - 1}. \tag{4.12}$$

We can further show that among all the flow of the network system satisfying that the expected terminal flow value is z, the optimal variance of $x(T)$ is

$$\begin{aligned} E(x^*(T) - z)^2 = {} & \frac{\bar{P}(0,\alpha)\bar{H}(0,\alpha)^2 + \theta}{1 - \theta - \bar{P}(0,\alpha)\bar{H}(0,\alpha)^2}\left[z - \frac{\bar{P}(0,\alpha)\bar{H}(0,\alpha)}{\bar{P}(0,\alpha)\bar{H}(0,\alpha)^2 + \theta}\hat{x}\right]^2 \\ & + \frac{\bar{P}(0,\alpha)\theta}{\bar{P}(0,\alpha)\bar{H}(0,\alpha)^2 + \theta}\hat{x}^2. \end{aligned} \tag{4.13}$$

Therefore, the minimum terminal variance is

$$E(x^*(T) - z)^2 = \frac{\bar{P}(0,\alpha)\theta}{\bar{P}(0,\alpha)\bar{H}(0,\alpha)^2 + \theta}\hat{x}^2 \geq 0 \tag{4.14}$$

with the minimum expected terminal flow of the system

$$z_{\min} = \frac{\bar{P}(0,\alpha)\bar{H}(0,\alpha)}{\bar{P}(0,\alpha)\bar{H}(0,\alpha)^2 + \theta}\hat{x} \tag{4.15}$$

and corresponding Lagrange multiplier $\lambda^*_{\min} = 0$.

(2) Assume that an efficient portfolio $u_1^*(t)$ is given by (4.11) corresponding to $z = z_1 > z_{\min}$. Then a control $u^*(t)$ is efficient if and only if there is a $\pi \geq 0$ such that

$$u^*(t) = (1 - \pi)u^*_{\min}(t) + \pi u_1^*(t). \tag{4.16}$$

where $t \in [0, T]$ and

$$u^*_{\min}(t) = -\big(\sigma(t, s_{kj})\sigma'(t, s_{kj})\big)^{-1} B'(t, s_{kj})\big[x - z_{\min}\bar{H}(t, k)\big]. \tag{4.17}$$

Assertion (2) is known as 'mutual fund theorem' in the financial market problems. In platoon control problems, this result offers a practical way of selecting the optimal flow controls so that the total platoon length can be as close to the designated value in the sense that the variance of the platoon length is minimized. Similarly, in platoon communication resource allocation problems, this strategy is optimal in the sense that the designated total throughput for the platoon communication network is most efficiently used.

4.2 Inclusion of transient states

In this section, we consider the case in which the Markov chain has transient states. We assume that

$$\tilde{Q} = \begin{pmatrix} \tilde{Q}_1 & 0 \\ \tilde{Q}_0 & \tilde{Q}_* \end{pmatrix},$$

where $\tilde{Q}_1 = \operatorname{diag}\{\tilde{Q}^1, \tilde{Q}^2, \ldots, \tilde{Q}^l\}$, $\tilde{Q}_0 = (\tilde{Q}^1_*, \ldots, \tilde{Q}^l_*)$. For each $k = 1, 2, \ldots, l$, $\tilde{Q}^k$ is a generator with dimension $m_k \times m_k$, $\tilde{Q}_* \in \mathbb{R}^{m_* \times m_*}$, $\tilde{Q}^k_* \in \mathbb{R}^{m_* \times m_k}$ and $m_1 + m_2 + \cdots + m_* = m$. The state space of the underlying Markov chain is given by $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \cdots \cup \mathcal{M}_* = \{s_{11}, \ldots, s_{1m_1}, \ldots, s_{l1}, \ldots, s_{lm_l}, s_{*1}, \ldots, s_{*m_*}\}$, where $\mathcal{M}_* = \{s_{*1}, s_{*2}, \ldots, s_{*m_*}\}$ consists of the transient states. Suppose that for $k = 1, 2, \ldots, l$, $\tilde{Q}^k$ are irreducible, and $\tilde{Q}_*$ is Hurwitz, i.e., all of its eigenvalues have negative real parts.

Let

$$\hat{Q} = \begin{pmatrix} \hat{Q}^{11} & \hat{Q}^{12} \\ \hat{Q}^{21} & \hat{Q}^{22} \end{pmatrix},$$

where $\hat{Q}^{11} \in \mathbb{R}^{(m-m_*)\times(m-m*)}$, $\hat{Q}^{12} \in \mathbb{R}^{(m-m_*)\times m_*}$, $\hat{Q}^{21} \in \mathbb{R}^{m_*\times(m-m_*)}$ and $\hat{Q}^{22} \in \mathbb{R}^{m_*\times m_*}$. We define

$$\bar{Q}_* = \operatorname{diag}(\mu^1, \ldots, \mu^l)(\hat{Q}^{11}\tilde{\mathbb{1}} + \hat{Q}^{12}(a_{m_1}, a_{m_2}, \ldots, a_{m_l}))$$

with $\tilde{\mathbb{1}} = \operatorname{diag}(\mathbb{1}_{m_1}, \ldots, \mathbb{1}_{m_l})$, $\mathbb{1}_{m_j} = (1, \ldots, 1)' \in \mathbb{R}^{m_j \times 1}$ and, for $k = 1, \ldots, l$,

$$a_{m_k} = (a_{m_k,1}, \ldots, a_{m_k,m_*})' = -\tilde{Q}_*^{-1}\tilde{Q}^k_* \mathbb{1}_{m_k}.$$

Let ξ be a random variable uniformly distributed on $[0, 1]$ that is independent of $\alpha^\varepsilon(\cdot)$. For each $j = 1, 2, \ldots, m_*$, define an integer-valued random variable ξ_j by

$$\xi_j = I_{\{0 \le \xi \le a_{m_1 j}\}} + 2I_{\{a_{m_1 j} < \xi \le a_{m_1 j} + a_{m_2 j}\}} + \cdots + lI_{\{a_{m_1 j} + \cdots + a_{m_{l-1} j} < \xi \le 1\}}.$$

Now define the aggregated process $\bar{\alpha}^\varepsilon(\cdot)$ by

$$\bar{\alpha}^\varepsilon(t) = \begin{cases} k, & \text{if } \alpha^\varepsilon(t) \in \mathcal{M}_k, \\ \xi_j, & \text{if } \alpha^\varepsilon(t) = s_{*j}. \end{cases}$$

Note the state space of $\bar{\alpha}^\varepsilon(t)$ is $\bar{\mathcal{M}} = \{1, 2, \ldots, l\}$ and $\bar{\alpha}^\varepsilon(\cdot) \in D[0, T]$. In addition,

$$P(\bar{\alpha}^\varepsilon(t) = i | \alpha^\varepsilon(t) = s_{*j}) = a_{m_i j}.$$

Then according to [8] (Theorem 4.2), $\bar{\alpha}^\varepsilon(\cdot)$ converges weakly to $\bar{\alpha}(\cdot)$ such that $\bar{\alpha}(\cdot) \in \{1, 2, \ldots, l\}$ is a Markov chain generated by $\bar{Q}_*$.

THEOREM 4.6. As $\varepsilon \to 0$, we have $P^\varepsilon(s, s_{kj}) \to \bar{P}(s, k)$ and $H^\varepsilon(s, s_{kj}) \to \bar{H}(s, k)$, for $k = 1, 2, \ldots, l$, $j = 1, 2, \ldots, m_k$, $P^\varepsilon(s, s_{*j}) \to \bar{P}_*(s, j)$ and $H^\varepsilon(s, s_{*j}) \to \bar{H}_*(s, j)$, for $j = 1, 2, \ldots, m_*$ uniformly on $[0, T]$ where

$$\bar{P}_*(s, j) = a_{m_1 j}\bar{P}(s, 1) + \cdots + a_{m_l j}\bar{P}(s, l),$$

$$\bar{H}_*(s, j) = a_{m_1 j}\bar{H}(s, 1) + \cdots + a_{m_l j}\bar{H}(s, l)$$

and $\bar{P}(s, k)$ and $\bar{H}(s, k)$ are the unique solutions to the following equations. For $k = 1, 2, \ldots, l$,

$$\dot{\bar{P}}(t, k) = (\bar{\rho}(t, k) - 2\bar{r}(t, k))\bar{P}(t, k) - \bar{Q}_*\bar{P}(t, \cdot)(k), \quad \bar{P}(T, k) = 1 \tag{4.18}$$

and

$$\begin{aligned} \dot{\bar{H}}(t, k) &= \bar{r}(t, k)\bar{H}(t, k) - \frac{1}{\bar{P}(t, k)}\bar{Q}_*\bar{P}(t, \cdot)\bar{H}(t, \cdot)(k) + \frac{\bar{H}(t, k)}{\bar{P}(t, k)}\bar{Q}_*\bar{P}(t, \cdot)(k), \\ \bar{H}(T, k) &= 1. \end{aligned} \tag{4.19}$$

The convergence of P^ε and H^ε leads to $v^\varepsilon(t, s_{kj}, x) \to \bar{v}(t, k, x)$, for $k = 1, 2, \ldots, l$, $j = 1, 2, \ldots, m_k$, $v^\varepsilon(t, s_{*j}, x) \to v_*(t, j, x)$ for $j = 1, 2, \ldots, m_*$, where

$$v_*(t, j, x) = a_{m_1 j}\bar{v}(t, 1, x) + \cdots + a_{m_l j}\bar{v}(t, l, x)$$

and $\bar{v}(t, k, x) = \bar{P}(t, k)(x + (\lambda - z)\bar{H}(t, k))^2$. The control set for the limit problem is the same as that for recurrent case and is given by

$$\mathcal{U} = \left\{U = (U^1, U^2, \ldots, U^l) : U^k = (u^{k1}, u^{k2}, \ldots, u^{km_k}),\ u^{kj} \in \mathbb{R}^{d_1}\right\}.$$

Then the corresponding limit problem is

$$\mathrm{d}x(t) = f(x(t), \bar{\alpha}(t), U(t))\,\mathrm{d}t + \sum_{i=1}^{d} g_i(t, \bar{\alpha}(t), U(t))\,\mathrm{d}w_i(t),$$

where $\bar{\alpha}(\cdot) \in \{1, 2, \ldots, l\}$ is a Markov chain generated by $\bar{Q}_*$ with $\bar{\alpha}(0) = \alpha$. The optimal control for this limit problem is

$$U^*(t) = \left(U^{1*}(t, x), U^{2*}(t, x), \ldots, U^{l*}(t, x)\right)$$

with

$$U^{k*}(t,x) = \left(u^{k1*}(t,x), u^{k2*}(t,x), \ldots, u^{km_k*}(t,x)\right)$$

and

$$u^{kj*}(t,x) = -\left(\sigma(t,s_{kj})\sigma'(t,s_{kj})\right)^{-1}B'(t,s_{kj})\left[x + (\lambda - z)\bar{H}(t,k)\right].$$

Using such controls, we construct

$$u^{\varepsilon}(t,\alpha^{\varepsilon}(t),x) = \sum_{k=1}^{l}\sum_{j=1}^{m_k} I_{\{\alpha^{\varepsilon}(t)=s_{kj}\}}u^{kj*}(t,x) + \sum_{j=1}^{m_*} I_{\{\alpha^{\varepsilon}(t)=s_{*j}\}}u^{*j*}(t,x) \tag{4.20}$$

for the original problem where $u^{*j*}(s,x) = -(\sigma(t,s_{*j})\sigma'(t,s_{*j}))^{-1}B'(t,s_{*j})$ $[x + (\lambda - z)\bar{H}_*(s,j)]$.

Proof. Following the proof of Theorem 4.1 to (4.3), we have for $s \in [0,T]$, $\tilde{Q}^k P^0(s,\cdot)\times (s_{kj}) = 0$ for $k = 1,2,\ldots,l$, $j = 1,2,\ldots,m_k$

$$\left(\tilde{Q}_*^1, \ldots, \tilde{Q}_*^l, \tilde{Q}_*\right)\left(P^0(s,s_{11}), \ldots, P^0(s,s_{1m_1}), \ldots,\right.$$
$$\left. P^0(s,s_{l1}), \ldots, P^0(s,s_{lm_l}), P^0(s,s_{*1}), \ldots, P^0(s,s_{*m*})\right)' = 0.$$

The irreducibility of $\tilde{Q}^k$ for any k implies

$$(P^0(s,s_{k1}), \ldots, P^0(s,s_{km_k}))' = P^0(s,k)\mathbb{1}_{m_k}.$$

Let $P_*(s) = (P^0(s,s_{*1}), \ldots, P^0(s,s_{*m*}))'$, we have

$$\tilde{Q}_*^1\mathbb{1}_{m_1}P^0(s,1) + \cdots + \tilde{Q}_*^l\mathbb{1}_{m_l}P^0(s,l) + \tilde{Q}_*P_*(s) = 0.$$

Here,

$$P_*(s) = -\tilde{Q}_*^{-1}\left(\tilde{Q}_*^1\mathbb{1}_{m_1}P^0(s,1) + \cdots + \tilde{Q}_*^l\mathbb{1}_{m_l}P^0(s,l)\right) = a_{m_1}P^0(s,1) + \cdots + a_{m_l}P^0(s,l).$$

Then $P_*(s) \in \mathbb{R}^{m_*}$ and its jth component is $P_*(s,j)$. The rest of the proof is similar to that of Theorem 4.1, except replacing $\bar{Q}$ by $\bar{Q}_*$. □

THEOREM 4.7. The control $u^{\varepsilon}(t)$ defined in (4.20) is nearly optimal in that $\lim_{\varepsilon\to 0}|J^{\varepsilon}(\alpha,x,u^{\varepsilon}(\cdot)) - v^{\varepsilon}(\alpha,x)| = 0$.

Proof. The proof is similar to that of Theorem 4.4 with the use of the estimate $E|\int_0^t I_{\{\alpha^{\varepsilon}(s)=s_{*j}\}}\,ds|^2 \to 0$ as $\varepsilon \to 0$ from [8] (Theorem 3.1). □

5. Further remarks

This work has focused on the near-optimal controls for non-definite control problems. Our primary motivation stems from networked systems. Our approach provides a systematic approach to reducing the complexity of the underlying system. In lieu of treating the large dimensional systems directly, we solve a set of limit Riccati equations that have much smaller dimensions. Using the limit problem as a guide to design controls for the original system leads to near-optimal controls of the original system. Although the paper is devoted to platoon controls, the results can be readily applied to the portfolio optimization in financial engineering. Future research efforts can be directed to the study of non-definite control problems in the hybrid systems, in which the Markov chain $\alpha(t)$ is a hidden process and the Wonham filter is involved. A combination of weak convergence methods ([2] and [7]) and optimal control is another worthwhile effort. More thoughts and further considerations are needed.

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References

[1] S. Asmussen and M. Taksar, *Controlled diffusion models for optimal dividend pay-out*, Insur. Math. Econ. 20 (1997), pp. 1–15.
[2] H.J. Kushner, *Approximation and Weak Convergence Methods for Random Processes, with Application to Stochastic Systems Theory*, MIT Press, Cambridge, MA, 1984.
[3] D. Li and W.L. Ng, *Optimal dynamic portfolio selection: Multi-period mean–variance formulation*, Math. Finance 10 (2000), pp. 387–406.
[4] Y.J. Liu, G. Yin, and X.Y. Zhou, *Near-optimal controls of random-switching LQ with indefinite control weight*, Automatica 41 (2005), pp. 1063–1070.
[5] H. Markowitz, *Portfolio selection*, J. Finance 7 (1952), pp. 77–91.
[6] H.A. Simon and A. Ando, *Aggregation of variables in dynamic systems*, Econometrica 29 (1961), pp. 111–138.
[7] G. Yin and Q. Zhang, *Continuous-Time Markov Chains and Applications: A Singular Perturbation Approach*, Springer-Verlag, New York, 1998.
[8] G. Yin, Q. Zhang, and G. Badowski, *Asymptotic properties of a singularly perturbed Markov chain with inclusion of transient states*, Ann. Appl. Probab. 10 (2000), pp. 549–572.
[9] Q. Zhang and G. Yin, *On nearly optimal controls of hybrid LQG problems*, IEEE Trans. Automat. Control 44 (1999), pp. 2271–2282.
[10] X.Y. Zhou and D. Li, *Continuous-time mean–variance portfolio selection: A stochastic LQ framework*, Appl. Math. Optim. 42 (2000), pp. 19–33.
[11] X.Y. Zhou and G. Yin, *Markowitz's mean–variance portfolio selection with regime switching: A continuous time model*, SIAM J. Control Optim. 42 (2003), pp. 1466–1482.